
TB 13.4 Polytropes 01-01-18

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Initialization: Be sure the file *NTGUtilityFunctions.m* is in the same directory as that from which this notebook was loaded. Then execute the cell immediately below by mousing left on the cell bar to the right of that cell and then typing “shift” + “enter”. Respond “Yes” in response to the query to evaluate initialization cells.

```
SetDirectory[NotebookDirectory[]];  
(* set directory where source files are located *)  
Get["NTGUtilityFunctions.m"]; (* Load utilities package *)
```

Purpose

This is the first in a series of notebooks in which I work through material and exercises in the magisterial new book *Modern Classical Physics* by Kip S. Thorne and Roger D. Blandford. If you are a physicist of any ilk, BUY THIS BOOK. You will learn from a close reading and from solving the exercises.

In this series, I am mostly using Mathematica to solve exercises in Chapter 13 *Foundations of Fluid Dynamics*. Although I was a plasma physicist and published numerous articles in the AIP Journal *Physics of Fluids*, I never actually had a formal course in fluids. Guided by my graduate teachers, I moved quickly into consideration of waves and instabilities in plasmas and didn't have time to firm up my foundations in fluid theory. Working through Chapter 13 gave me an opportunity to consider topics in the physics of fluids that I had never encountered in my researches. I mention that while Thorne and Blandford cover topics in relativistic fluids, I limit my treatment to classical fluids.

When possible I am copying problem statements from material freely available on the web (there were pdf versions of *Modern Classical Physics* used for graduate courses during the years before publication of the book). In some cases, however, the problem statement in the new book differ from the problem statement available on the web. In that case, I copied an image of the problem from my copy of the book so that I would not have to paraphrase the problem.

Exercise 13.4 ****Example: Polytropes — The Power of Dimensionless Variables*

When dealing with differential equations describing a physical system, it is often helpful to convert to dimensionless variables. Polytropes (nonrotating, spherical fluid bodies with the polytropic equation of state $P = K\rho^{1+1/n}$) are a nice example.

- (a) Combine the two equations of stellar structure (13.16) to obtain a single second-order differential equation for P and ρ as functions of r .

- (b) In this equation set $P = K\rho^{1+1/n}$ to obtain a nonlinear, second-order differential equation for $\rho(r)$.
- (c) It is helpful to change dependent variables from $\rho(r)$ to some other variable, call it $\theta(r)$, so chosen that the quantity being differentiated is linear in θ and the only θ -nonlinearity is in the driving term. Show that choosing $\rho \propto \theta^n$ achieves this.
- (d) It is helpful to choose the proportionality constant in $\rho \propto \theta^n$ in such a way that θ is dimensionless and takes the value 1 at the polytrope's center and 0 at its surface. This is achieved by setting

$$\rho = \rho_c \theta^n, \quad (13.25a)$$

where ρ_c is the polytrope's (possibly unknown) central density.

- (e) Similarly, it is helpful to make the independent variable r dimensionless by setting $r = a\xi$, where a is a constant with dimensions of length. The value of a should be chosen wisely, so as to simplify the differential equation as much as possible. Show that the choice

$$r = a\xi, \quad \text{where } a = \left[\frac{(n+1)K\rho_c^{(1/n-1)}}{4\pi G} \right]^{1/2}, \quad (13.25b)$$

brings the differential equation into the form

$$\frac{1}{\xi^2} \frac{d}{d\xi} \xi^2 \frac{d\theta}{d\xi} = -\theta^n. \quad (13.26)$$

This is called the *Lane-Emden equation of stellar structure*, after Jonathan Homer Lane and Jacob Robert Emden, who introduced and explored it near the end of the 19th century. There is an extensive literature on solutions of the Lane-Emden equation; see, especially, Chap. 4 of Chandrasekhar (1939) and Sec. 3.3 of Shapiro and Teukolsky (1984).

- (f) Explain why the Lane-Emden equation must be solved subject to the following boundary conditions (where $\theta' \equiv d\theta/d\xi$):

$$\theta = \theta' = 0 \text{ at } \xi = 0. \quad (13.27)$$

- (g) One can integrate the Lane-Emden equation, numerically or analytically, outward from $\xi = 0$ until some radius ξ_1 at which θ (and thus also ρ and P) goes to zero. That is the Polytrope's surface. Its physical radius is then $R = a\xi_1$, and its mass is $M = \int_0^R 4\pi\rho r^2 dr$, which is readily shown to be $M = 4\pi a^3 \rho_c \xi_1^2 |\theta'(\xi_1)|$; i.e., using the value (13.25b) of a :

$$R = \left[\frac{(n+1)K}{4\pi G} \right]^{1/2} \rho_c^{(1-n)/2n} \xi_1, \quad M = 4\pi \left[\frac{(n+1)K}{4\pi G} \right]^{3/2} \rho_c^{(3-n)/2n} \xi_1^2 |\theta'(\xi_1)|, \quad (13.28a)$$

$$\text{whence } M = 4\pi R^{(3-n)/(1-n)} \left[\frac{(n+1)K}{4\pi G} \right]^{n/(n-1)} \xi_1^2 |\theta'(\xi_1)|. \quad (13.28b)$$

- (h) Whenever one converts a problem into dimensionless variables that satisfy some differential or algebraic equation(s), and then expresses physical quantities in terms of the dimensionless variables, the resulting expressions describe how the physical quantities scale with each other. As an example: Jupiter and Saturn are both comprised of a H-He fluid that is well approximated by a polytrope of index $n = 1$, $P = K\rho^2$, with the same constant K . Use the information that $M_J = 2 \times 10^{27}\text{kg}$, $R_J = 7 \times 10^4\text{km}$, $M_S = 6 \times 10^{26}\text{kg}$, to estimate the radius of Saturn. For $n = 2$, the Lane-Emden equation has a simple analytical solution, $\theta = \sin(\xi)/\xi$. Compute the central densities of Jupiter and Saturn.

Analysis and solution

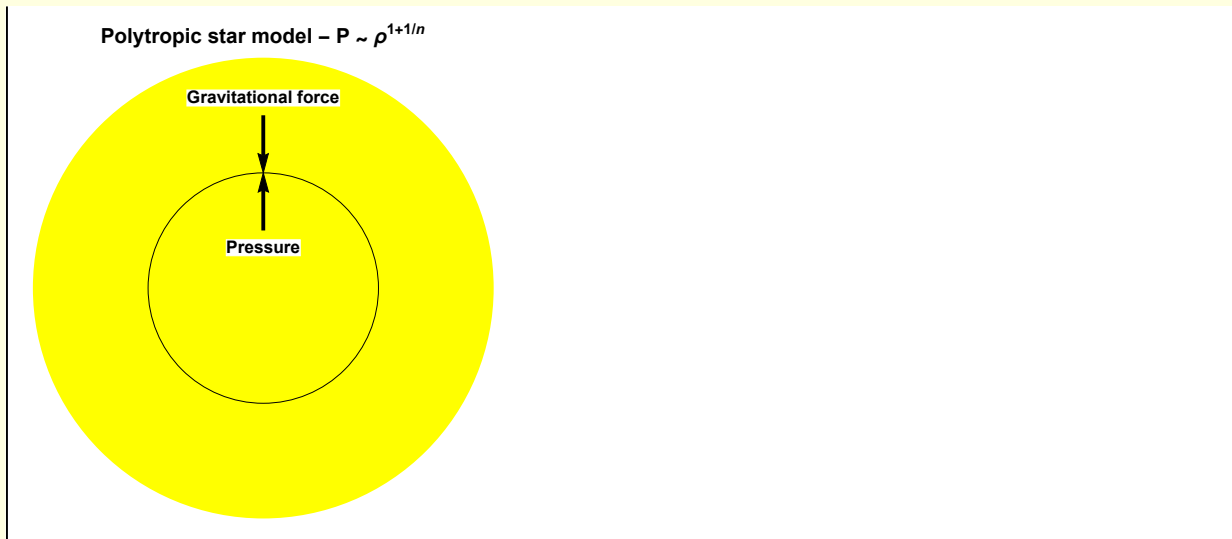
I use the Mathematica Notation package

<< Notation`

Symbolize[\tilde{r}];

Symbolize[ρ_c];

Symbolize[ξ_s];

Part (a) Ordinary differential equation linking $P(r)$ and $\rho(r)$ 

The radial pressure balance equation for a gravitating star is

$$\text{wa}[1] = \text{D}[P[r], r] == - \frac{G m[r] \rho[r]}{r^2}$$

$$P'[r] == - \frac{G m[r] \rho[r]}{r^2}$$

To obtain an ode directly relating pressure $P(r)$ and mass density $\rho(r)$, it is necessary to eliminate $m(r)$, the mass contained in a sphere of radius r

$$\text{wa}[2] = m[r] == \text{Integrate}[4 \pi \rho[\tilde{r}] \tilde{r}^2, \{\tilde{r}, 0, r\}]$$

$$m[r] == \int_0^r 4 \pi \tilde{r}^2 \rho[\tilde{r}] d\tilde{r}$$

Differentiate the pressure balance equation

$$\text{wa}[3] = \text{MapEqn}[\text{D}[\#, r] \&, \text{wa}[1]]$$

$$P''[r] == \frac{2 G m[r] \rho[r]}{r^3} - \frac{G \rho[r] m'[r]}{r^2} - \frac{G m[r] \rho'[r]}{r^2}$$

Use the definition of $m(r)$ to obtain an explicit expression for $m'(r)$ only involving $\rho(r)$

$$\text{wa}[4] = \text{MapEqn}[\text{D}[\#, r] \&, \text{wa}[2]]$$

$$m'[r] == 4 \pi r^2 \rho[r]$$

Also, the original pressure balance equation yields another expression for $m(r)$

$$\text{wa}[5] = \text{Solve}[\text{wa}[1], m[r]] \llbracket 1, 1 \rrbracket$$

$$m[r] \rightarrow -\frac{r^2 P'[r]}{G \rho[r]}$$

With these expressions for $m(r)$ and $m'(r)$ the desired ode linking $P(r)$ and $\rho(r)$ is obtained

$$\text{wa}[6] = \text{wa}[3] /. \text{Solve}[\text{wa}[4], m'[r]] \llbracket 1, 1 \rrbracket /. \text{wa}[5]$$

$$P''[r] == -4 G \pi \rho[r]^2 - \frac{2 P'[r]}{r} + \frac{P'[r] \rho'[r]}{\rho[r]}$$

Then

$$\text{wa}["\text{final}"] = P''[r] == -4 G \pi \rho[r]^2 - \frac{2 P'[r]}{r} + \frac{P'[r] \rho'[r]}{\rho[r]}$$

$$P''[r] == -4 G \pi \rho[r]^2 - \frac{2 P'[r]}{r} + \frac{P'[r] \rho'[r]}{\rho[r]}$$

The differential equation linking pressure and density is

$$P''[r] == -4 G \pi \rho[r]^2 - \frac{2 P'[r]}{r} + \frac{P'[r] \rho'[r]}{\rho[r]}$$

Part (b) ODE for $\rho(r)$ assuming polytropic equation of state

The polytropic equation of state is

$$\text{wb}[1] = P[r] == K \rho[r]^{1+1/n}$$

$$P[r] == K \rho[r]^{1+\frac{1}{n}}$$

Use this in the result from part (a)

$$\text{wb}[2] = \text{wa}["\text{final}"] /. P \rightarrow \text{Function}[\{r\}, K \rho[r]^{1+1/n}]$$

$$\frac{K \left(1 + \frac{1}{n}\right) \rho[r]^{-1+\frac{1}{n}} \rho'[r]^2}{n} + K \left(1 + \frac{1}{n}\right) \rho[r]^{\frac{1}{n}} \rho''[r] ==$$

$$-4 G \pi \rho[r]^2 - \frac{2 K \left(1 + \frac{1}{n}\right) \rho[r]^{\frac{1}{n}} \rho'[r]}{r} + K \left(1 + \frac{1}{n}\right) \rho[r]^{-1+\frac{1}{n}} \rho'[r]^2$$

Perform some simplifying manipulations

wb[3] = StandardizeEqn[wb[2]]

$$4 G \pi \rho [r]^2 + \frac{2 K \left(1 + \frac{1}{n}\right) \rho [r]^{\frac{1}{n}} \rho' [r]}{r} -$$

$$K \left(1 + \frac{1}{n}\right) \rho [r]^{-1 + \frac{1}{n}} \rho' [r]^2 + \frac{K \left(1 + \frac{1}{n}\right) \rho [r]^{-1 + \frac{1}{n}} \rho' [r]^2}{n} + K \left(1 + \frac{1}{n}\right) \rho [r]^{\frac{1}{n}} \rho'' [r] == 0$$

wb[4] = (# / (K (1 + 1/n))) & /@wb[3][[1]] == 0

$$\frac{4 G \pi \rho [r]^2}{K \left(1 + \frac{1}{n}\right)} + \frac{2 \rho [r]^{\frac{1}{n}} \rho' [r]}{r} - \rho [r]^{-1 + \frac{1}{n}} \rho' [r]^2 + \frac{\rho [r]^{-1 + \frac{1}{n}} \rho' [r]^2}{n} + \rho [r]^{\frac{1}{n}} \rho'' [r] == 0$$

The differential equation for density, assuming a polytropic equation of state, is

wb["final"] =

$$\frac{4 G \pi \rho [r]^2}{K \left(1 + \frac{1}{n}\right)} + \frac{2 \rho [r]^{\frac{1}{n}} \rho' [r]}{r} - \rho [r]^{-1 + \frac{1}{n}} \rho' [r]^2 + \frac{\rho [r]^{-1 + \frac{1}{n}} \rho' [r]^2}{n} + \rho [r]^{\frac{1}{n}} \rho'' [r] == 0$$

$$\frac{4 G \pi \rho [r]^2}{K \left(1 + \frac{1}{n}\right)} + \frac{2 \rho [r]^{\frac{1}{n}} \rho' [r]}{r} - \rho [r]^{-1 + \frac{1}{n}} \rho' [r]^2 + \frac{\rho [r]^{-1 + \frac{1}{n}} \rho' [r]^2}{n} + \rho [r]^{\frac{1}{n}} \rho'' [r] == 0$$

$$\frac{4 G \pi \rho [r]^2}{K \left(1 + \frac{1}{n}\right)} + \frac{2 \rho [r]^{\frac{1}{n}} \rho' [r]}{r} - \rho [r]^{-1 + \frac{1}{n}} \rho' [r]^2 + \frac{\rho [r]^{-1 + \frac{1}{n}} \rho' [r]^2}{n} + \rho [r]^{\frac{1}{n}} \rho'' [r] == 0$$

Part (c) and part(d) Convenient form for the differential equation for $\rho(r)$

Introduce the dependent variable θ

wc[1] = wb["final"] /. $\rho \rightarrow \text{Function}[\{r\}, \rho_c \theta [r]^n]$

$$\frac{4 G \pi \rho_c^2 \theta [r]^{2n}}{K \left(1 + \frac{1}{n}\right)} + \frac{2 n \rho_c \theta [r]^{-1+n} (\rho_c \theta [r]^n)^{\frac{1}{n}} \theta' [r]}{r} +$$

$$n \rho_c^2 \theta [r]^{-2+2n} (\rho_c \theta [r]^n)^{-1 + \frac{1}{n}} \theta' [r]^2 - n^2 \rho_c^2 \theta [r]^{-2+2n} (\rho_c \theta [r]^n)^{-1 + \frac{1}{n}} \theta' [r]^2 +$$

$$(\rho_c \theta [r]^n)^{\frac{1}{n}} ((-1+n) n \rho_c \theta [r]^{-2+n} \theta' [r]^2 + n \rho_c \theta [r]^{-1+n} \theta'' [r]) == 0$$

where ρ_c is a parameter interpreted as the density as $r \rightarrow 0$.

Perform some simplifications

`wc[2] = Simplify[PowerExpand[#]] & /@ wc[1][[1]] == 0`

$$\frac{4 G n \pi \rho_c^2 \theta[r]^{2n}}{K + K n} + \frac{2 n \rho_c^{1+\frac{1}{n}} \theta[r]^n \theta'[r]}{r} + n \rho_c^{1+\frac{1}{n}} \theta[r]^{-1+n} \theta'[r]^2 - n^2 \rho_c^{1+\frac{1}{n}} \theta[r]^{-1+n} \theta'[r]^2 + n \rho_c^{1+\frac{1}{n}} \theta[r]^{-1+n} ((-1+n) \theta'[r]^2 + \theta[r] \theta''[r]) == 0$$

`wc[3] = (# / (n \rho_c^{1+1/n} \theta[r]^n)) & /@ wc[2][[1]] == 0 // Expand`

$$\frac{4 G \pi \rho_c^{1-\frac{1}{n}} \theta[r]^n}{K + K n} + \frac{2 \theta'[r]}{r} + \theta''[r] == 0$$

The differential equation for θ is

$$\text{wc["final"]} = \frac{4 G \pi \rho_c^{1-\frac{1}{n}} \theta[r]^n}{K + K n} + \frac{2 \theta'[r]}{r} + \theta''[r] == 0$$

$$\frac{4 G \pi \rho_c^{1-\frac{1}{n}} \theta[r]^n}{K + K n} + \frac{2 \theta'[r]}{r} + \theta''[r] == 0$$

$$\frac{4 G \pi \rho_c^{1-\frac{1}{n}} \theta[r]^n}{K + K n} + \frac{2 \theta'[r]}{r} + \theta''[r] == 0$$

Part (e) Convenient independent variable

Further simplify the ode for θ by introducing a radial scale factor

`wc[1] = wc["final"] /. \theta \to Function[{r}, \theta[r/a]] /. r \to a \xi`

$$\frac{4 G \pi \rho_c^{1-\frac{1}{n}} \theta[\xi]^n}{K + K n} + \frac{2 \theta'[\xi]}{a^2 \xi} + \frac{\theta''[\xi]}{a^2} == 0$$

`wc[2] = MapEqn[(# a^2) &, wc[1]] // Expand`

$$\frac{4 a^2 G \pi \rho_c^{1-\frac{1}{n}} \theta[\xi]^n}{K + K n} + \frac{2 \theta'[\xi]}{\xi} + \theta''[\xi] == 0$$

Choose the parameter a to have the form

$$\text{def}[a] = a == \left((n+1) \frac{K \rho_c^{-1+1/n}}{4 \pi G} \right)^{1/2}$$

$$a == \frac{\sqrt{\frac{K (1+n) \rho_c^{-1+\frac{1}{n}}}{G}}}{2 \sqrt{\pi}}$$

$$\text{we}[3] = \text{we}[2] /. (\text{def}[a] // \text{ER}) // \text{Simplify}$$

$$\theta[\xi]^n + \frac{2 \theta'[\xi]}{\xi} + \theta''[\xi] == 0$$

Note that

$$\text{we}[4] = \text{HoldForm}\left[\frac{1}{\xi^2} D[\xi^2 D[\theta[\xi], \xi], \xi]\right] == \frac{1}{\xi^2} D[\xi^2 D[\theta[\xi], \xi], \xi] // \text{Expand}$$

$$\frac{\partial_\xi (\xi^2 \partial_\xi \theta[\xi])}{\xi^2} == \frac{2 \theta'[\xi]}{\xi} + \theta''[\xi]$$

Then

$$\text{we}[5] = \text{we}[3] /. \text{Solve}[\text{we}[4], \theta''[\xi]] [[1, 1]] // \text{Expand}$$

$$\frac{\partial_\xi (\xi^2 \partial_\xi \theta[\xi])}{\xi^2} + \theta[\xi]^n == 0$$

This is the Lane-Emden equation of stellar structure.

$$\text{we}["\text{final}"] = \frac{\partial_\xi (\xi^2 \partial_\xi \theta[\xi])}{\xi^2} + \theta[\xi]^n == 0$$

$$\frac{\partial_\xi (\xi^2 \partial_\xi \theta[\xi])}{\xi^2} + \theta[\xi]^n == 0$$

$$\frac{\partial_\xi (\xi^2 \partial_\xi \theta[\xi])}{\xi^2} + \theta[\xi]^n == 0$$

Part (f) Derivation of boundary conditions

Recall the polytropic relationship between pressure and density

$$\text{wf}[1] = \text{wb}[1]$$

$$P[r] == K \rho[r]^{1+\frac{1}{n}}$$

and that

```
wf[2] = wf[1] /. ρ → Function[{r}, ρc θ[r]n] // PowerExpand
```

$$P[r] = K \rho_c^{1+\frac{1}{n}} \theta[r]^{(1+\frac{1}{n})n}$$

with ρ_c defined as the density at the center of the star ($r = 0$). In order for

```
wf[3] = wf[2] /. r → θ
```

$$P[\theta] = K \rho_c^{1+\frac{1}{n}} \theta[\theta]^{(1+\frac{1}{n})n}$$

to satisfy $P[\theta] = K \rho_c^{1+\frac{1}{n}}$, θ must satisfy the boundary condition $\theta(0) = 1$

```
wf[4] = wf[3] /. θ[θ] → 1
```

$$P[\theta] = K \rho_c^{1+\frac{1}{n}}$$

The second boundary condition follows from the starting equations

```
{wa[1], wa[2]}
```

$$\left\{ P'[r] = -\frac{Gm[r] \rho[r]}{r^2}, m[r] = \int_0^r 4\pi \tilde{r}^2 \rho[\tilde{r}] d\tilde{r} \right\}$$

The pressure gradient is proportional to $m(r)$ and $m(r) \rightarrow 0$ as $r \rightarrow 0$

Consider the relationship between pressure and density near the center of the star

```
wf[5] = MapEqn[Normal@Series[#, {r, 0, 1}] &, wf[1]]
```

$$P[\theta] + r P'[\theta] = K \rho[\theta]^{1+\frac{1}{n}} + \frac{K(1+n) r \rho[\theta]^{\frac{1}{n}} \rho'[\theta]}{n}$$

```
wf[6] = wf[5] /. ρ[θ] → ρc /. (wf[4] // ER)
```

$$K \rho_c^{1+\frac{1}{n}} + r P'[\theta] = K \rho_c^{1+\frac{1}{n}} + \frac{K(1+n) r \rho_c^{\frac{1}{n}} \rho'[\theta]}{n}$$

The pressure gradient is

```
wf[7] = Solve[wf[6], P'[\theta]] [[1, 1]] // Simplify
```

$$P'[\theta] \rightarrow \frac{K(1+n) \rho_c^{\frac{1}{n}} \rho'[\theta]}{n}$$

Note

$$\mathbf{wf}[8] = \rho'[r] \rightarrow (\rho'[r] /. \rho \rightarrow \text{Function}[\{r\}, \rho_c \theta[r]^n]) /. r \rightarrow \theta$$

$$\rho'[\theta] \rightarrow n \rho_c \theta[\theta]^{-1+n} \theta'[\theta]$$

$$\mathbf{wf}[9] = \mathbf{wf}[7] /. \mathbf{wf}[8]$$

$$P'[\theta] \rightarrow K (1+n) \rho_c^{\frac{1+1}{n}} \theta[\theta]^{-1+n} \theta'[\theta]$$

In order for the pressure gradient to vanish at the center of the star, we must have the boundary conditions $\theta'[\theta] = 0$. Physically, the pressure gradient must tend to zero because there is contained mass tends to zero as r tends to zero.

$$\mathbf{wf}[10] = \mathbf{wf}[9] /. \theta'[\theta] \rightarrow \theta$$

$$P'[\theta] \rightarrow \theta$$

which implies

$$\mathbf{wf}[11] = \theta'[\theta] \rightarrow \theta$$

$$\theta'[\theta] \rightarrow \theta$$

Part (g) Calculation of radius and mass of a polytropic star

When the Lane-Emden equation is integrated outward, there exists a point ξ_s for which $\theta(\xi_s) = 0$, and hence $P(R) = 0$ and $\rho(R) = 0$. That is the radius of the star. In terms of model parameters

$$\mathbf{wg}[1] = R == a \xi_s /. (\mathbf{def}[a] // \mathbf{ER})$$

$$R == \frac{\xi_s \sqrt{\frac{K (1+n) \rho_c^{-\frac{1+1}{n}}}{G}}}{2 \sqrt{\pi}}$$

Define a term that simplifies the algebra

$$\mathbf{def}[\kappa] = \kappa == \frac{(n+1) K}{4 \pi G}$$

$$\kappa == \frac{K (1+n)}{4 G \pi}$$

```
wg[2] = wg[1] /. Sol[def[x], K] // PowerExpand
```

$$R = \sqrt{\kappa} \xi_s \rho_c^{\frac{1}{n}} \left(-1 + \frac{1}{n}\right)$$

The mass M of the star can be calculated from

```
wg[3] = wa[2] /. r -> R /. m[R] -> M
```

$$M = \int_0^R 4 \pi \tilde{r}^2 \rho[\tilde{r}] d\tilde{r}$$

or

```
wg[4] = wg[3] /. rho -> Function[{r}, rho_c theta[r]^n]
```

$$M = \int_0^R 4 \pi \tilde{r}^2 \rho_c \theta[\tilde{r}]^n d\tilde{r}$$

This integral can be evaluated by using the Lane Emden equation. Change variables

$$wg[5] = M = 4 \pi \rho_c a^3 \int_0^{\xi_s} \xi^2 \theta[\xi]^n d\xi$$

$$M = 4 a^3 \pi \rho_c \int_0^{\xi_s} \xi^2 \theta[\xi]^n d\xi$$

Note that the LE equation can be solved for $\theta[\xi]^n$

```
thetaRule = (Solve[we[5] /. theta[xi]^n -> x, x][[1, 1]]) /. x -> theta[xi]^n
```

$$\theta[\xi]^n \rightarrow -\frac{\partial_\xi (\xi^2 \partial_\xi \theta[\xi])}{\xi^2}$$

```
wg[6] = wg[5] /. thetaRule
```

$$M = 4 a^3 \pi \rho_c \int_0^{\xi_s} -\xi^2 \frac{\partial_\xi (\xi^2 \partial_\xi \theta[\xi])}{\xi^2} d\xi$$

Note that the integrand forms a perfect differential $\partial_\xi (\xi^2 \partial_\xi \theta[\xi]) d\xi$. So

```
wg[7] = M = -4 a^3 pi rho_c ((xi^2 partial_xi theta[xi]) /. xi -> xi_s) - (xi^2 partial_xi theta[xi] /. xi -> 0)
```

$$M = -4 a^3 \pi \xi_s^2 \rho_c \theta'[\xi_s]$$

Finally, note that θ is decreasing from $\theta(0) = 1$ to $\theta(\xi_s) = 0$, so

```
wg[8] = wg[7] /.  $\theta'[\xi_s] \rightarrow -\text{Abs}[\theta'[\xi_s]]$  /. (def[a] // ER) // PowerExpand
```

$$M = \left(\kappa^{3/2} (1+n)^{3/2} \xi_s^2 \rho_c^{1+\frac{3}{2}(-1+\frac{1}{n})} \text{Abs}[\theta'[\xi_s]] \right) / (2 G^{3/2} \sqrt{\pi})$$

```
wg[9] = wg[8] /. Sol[def[k], K] // Simplify[#, {G > 0, n > 0}] &
```

$$M = 4 \pi \kappa^{3/2} \xi_s^2 \rho_c^{\frac{1}{2} + \frac{3}{2n}} \text{Abs}[\theta'[\xi_s]]$$

Eliminate ρ_c by using the equation for R

```
wg[10] = Sol[wg[2],  $\rho_c$ ]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\rho_c \rightarrow \left(\frac{R}{\sqrt{\kappa} \xi_s} \right)^{-\frac{2n}{-1+n}}$$

```
wg[11] = wg[9] /. wg[10] // Simplify[#, {R > 0,  $\kappa > 0$ ,  $\xi_s > 0$ }] &
```

$$M = 4 \pi \kappa^{3/2} \left(\left(\frac{R}{\sqrt{\kappa} \xi_s} \right)^{-\frac{2n}{-1+n}} \right)^{-\frac{-3+n}{2n}} \xi_s^2 \text{Abs}[\theta'[\xi_s]]$$

```
wg[12] = wg[11] // PowerExpand // Simplify
```

$$M = 4 \pi R^{\frac{-3+n}{-1+n}} \kappa^{\frac{n}{-1+n}} \xi_s^{\frac{1+n}{-1+n}} \text{Abs}[\theta'[\xi_s]]$$

In terms of model parameters the mass of M of a Lane-Emden star is

$$\text{wg["finalR"]} = R = \sqrt{\kappa} \xi_s \rho_c^{\frac{1}{2}(-1+\frac{1}{n})}$$

$$R = \sqrt{\kappa} \xi_s \rho_c^{\frac{1}{2}(-1+\frac{1}{n})}$$

$$\text{wg["finalM"]} = M = 4 \pi \kappa^{3/2} \xi_s^2 \rho_c^{\frac{1}{2} + \frac{3}{2n}} \text{Abs}[\theta'[\xi_s]]$$

$$M = 4 \pi \kappa^{3/2} \xi_s^2 \rho_c^{\frac{1}{2} + \frac{3}{2n}} \text{Abs}[\theta'[\xi_s]]$$

Also

$$\text{wg["finalMR"]} = M == 4 \pi R^{\frac{-3+n}{-1+n}} \kappa^{\frac{n}{-1+n}} \xi_S^{\frac{1+n}{-1+n}} \text{Abs}[\theta'[\xi_S]]$$

$$M == 4 \pi R^{\frac{-3+n}{-1+n}} \kappa^{\frac{n}{-1+n}} \xi_S^{\frac{1+n}{-1+n}} \text{Abs}[\theta'[\xi_S]]$$

$$M == 4 \pi R^{\frac{-3+n}{-1+n}} \kappa^{\frac{n}{-1+n}} \xi_S^{\frac{1+n}{-1+n}} \text{Abs}[\theta'[\xi_S]]$$

Part (h) Jupiter and Saturn (n = 1 polytrope model)

For n = 1, the equation is state is quadratic

$$\text{wh}[1] = \text{wf}[1] /. n \rightarrow 1$$

$$P[r] == K \rho[r]^2$$

The derived expression for M is

$$\text{wh}[2] = \text{wg["finalM"]} /. \text{Sol}[\text{def}[\kappa], \kappa] /. n \rightarrow 1$$

$$M == \left(\frac{K}{G}\right)^{3/2} \sqrt{\frac{2}{\pi}} \xi_S^2 \rho_c \text{Abs}[\theta'[\xi_S]]$$

In Appendix I, I determine that for n = 1, $\xi_S = \pi$ and $\theta'(\pi) = -1/\pi$

$$\text{wh}[3] = \text{wh}[2] /. \xi_S \rightarrow \pi /. \theta'[\pi] \rightarrow -1/\pi$$

$$M == \left(\frac{K}{G}\right)^{3/2} \sqrt{2\pi} \rho_c$$

For Jupiter and Saturn, it is specified that the constant K is the same.

$$\text{wh}[4] = \{\text{wh}[3] /. \{M \rightarrow M_J, \rho_c \rightarrow \rho_{cJ}\}, \text{wh}[3] /. \{M \rightarrow M_S, \rho_c \rightarrow \rho_{cS}\}\}$$

$$\{M_J == \left(\frac{K}{G}\right)^{3/2} \sqrt{2\pi} \rho_{cJ}, M_S == \left(\frac{K}{G}\right)^{3/2} \sqrt{2\pi} \rho_{cS}\}$$

$$\text{wh}[5] = \text{Simplify}[\text{Eliminate}[\text{wh}[4], K], G > 0]$$

$$M_S \rho_{cJ} == M_J \rho_{cS}$$

For R

$$\text{wh}[6] = \text{wg}["\text{finalR}"] /. \text{Sol}[\text{def}[\kappa], \kappa] /. n \rightarrow 1 /. \xi_s \rightarrow \pi /. \theta'[\pi] \rightarrow -1/\pi$$

$$R = \sqrt{\frac{K}{G}} \sqrt{\frac{\pi}{2}}$$

Note that for $n = 1$, R depends only on K and, by this formula, the radii of Jupiter and Saturn should be the same. For $n = 1$, I also note that the Lane-Emden equation is linear.

However, from this formula I can calculate K

$$\text{wh}[7] = \text{wh}[6] /. R \rightarrow R_J /. \{R_J \rightarrow 7 \times 10^4 \times 10^3 \text{ m}\} /. G \rightarrow 6.67 \times 10^{-11} \text{ m}^3 / (\text{kg sec}^2)$$

$$70000000 \text{ m} = 153461. \sqrt{\frac{K \text{ kg sec}^2}{\text{m}^3}}$$

$$\text{wh}[8] = \text{Sol}[\text{wh}[7], K]$$

$$K \rightarrow \frac{208066. \text{ m}^5}{\text{kg sec}^2}$$

Then,

$$\text{wh}[9] = \text{wh}[4] /. \text{wh}[8] /. G \rightarrow 6.67 \times 10^{-11} \text{ m}^3 / (\text{kg sec}^2) // \text{PowerExpand}$$

$$\{M_J = 4.36721 \times 10^{23} \text{ m}^3 \rho_{cJ}, M_S = 4.36721 \times 10^{23} \text{ m}^3 \rho_{cS}\}$$

and

$$\text{wh}[10] = \{\text{Sol}[\text{wh}[9][1], \rho_{cJ}], \text{Sol}[\text{wh}[9][2], \rho_{cS}]\}$$

$$\{\rho_{cJ} \rightarrow \frac{2.28979 \times 10^{-24} M_J}{\text{m}^3}, \rho_{cS} \rightarrow \frac{2.28979 \times 10^{-24} M_S}{\text{m}^3}\}$$

The masses of Jupiter and Saturn are given, so

$$\text{wh}[11] = \text{wh}[10] /. \{M_J \rightarrow 2 \times 10^{27} \text{ kg}, M_S \rightarrow 6 \times 10^{26} \text{ kg}\}$$

$$\{\rho_{cJ} \rightarrow \frac{4579.58 \text{ kg}}{\text{m}^3}, \rho_{cS} \rightarrow \frac{1373.87 \text{ kg}}{\text{m}^3}\}$$

At this point, I don't see how to "estimate" the radius of Saturn without introducing additional physics beyond the Lane-Emden model with $n = 1$.

I looked at the paper http://astro.cornell.edu/academics/courses/astro6570/Outer_planet_interiors.pdf but still don't see what is required to "estimate" R_S .

Appendix I Lane-Emden for special case $n = 1$

```
wI[1] = we["final"] // ReleaseHold // Expand
```

$$\theta[\xi]^n + \frac{2\theta'[\xi]}{\xi} + \theta''[\xi] == 0$$

```
wI[2] = wI[1] /. n -> 1
```

$$\theta[\xi] + \frac{2\theta'[\xi]}{\xi} + \theta''[\xi] == 0$$

```
wI[3] = DSolve[{wI[2], \theta[0] == 1, \theta'[0] == 0}, \theta[\xi], \xi][[1, 1]]
```

Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is $i(2iC[1] + C[2]) == 0$.

Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is $2iC[1] + C[2] == 0$.

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\theta[\xi] \rightarrow -\frac{i e^{-i \xi} (-1 + e^{2 i \xi})}{2 \xi}$$

```
wI[4] = wI[3] // ExpToTrig // Simplify // RE
```

$$\theta[\xi] == \frac{\text{Sin}[\xi]}{\xi}$$

Since $\xi = \xi_s$ when $\theta(\xi_s) = 0$, I note that

```
wI[5] = \xi_s == \pi
```

$$\xi_s == \pi$$

```
wI[6] = MapEqn[D[#, \xi] &, wI[4]]
```

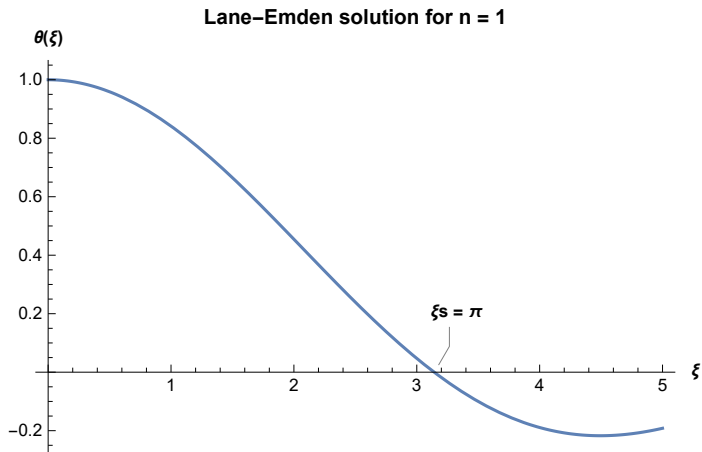
$$\theta'[\xi] == \frac{\text{Cos}[\xi]}{\xi} - \frac{\text{Sin}[\xi]}{\xi^2}$$

```
wI[7] = wI[6] /. \xi -> \pi
```

$$\theta'[\pi] == -\frac{1}{\pi}$$

```

Module[{root, lab, labCurve},
  root = FindRoot[ $\frac{\text{Sin}[\xi]}{\xi} == 0$ , { $\xi$ , 3}][[1, 2]];
  lab = St1["Lane-Emden solution for n = 1"];
  labCurve = St1@StringForm[" $\xi s = \pi$ ",  $\pi$ ];
  Plot[Callout[ $\frac{\text{Sin}[\xi]}{\xi}$ , labCurve,  $\pi$ ], { $\xi$ , 0, 5},
    AxesLabel -> {St1[" $\xi$ "], St1[" $\theta(\xi)$ 
```



Visualization

```
Module[{O = {0, 0}, a = 1, r = 0.5, star, circle, PVec, GVec, lab},
  PVec = {Directive[Black, Thick],
    Arrow[{{0, 0.25}, {0, 0.5}}], Text[Style["Pressure", Bold], {0, 0.175}]}];
  GVec = {Directive[Black, Thick], Arrow[{{0, 0.75}, {0, 0.5}}],
    Text[Style["Gravitational force", Bold], {0, 0.825}]}];
  circle = {Black, Circle[0, r]};
  star = {Yellow, Disk[0, a]};
  lab = Stl@StringForm["Polytropic star model -  $P \sim \rho^{1+1/n}$ "];
  Graphics[{star, circle, PVec, GVec}, PlotLabel -> lab, ImageSize -> 250]
```

