
TB 13.6 Binary Stars 01-10-18

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Initialization: Be sure the file *NTGUtilityFunctions.m* is in the same directory as that from which this notebook was loaded. Then execute the cell immediately below by mousing left on the cell bar to the right of that cell and then typing “shift” + “enter”. Respond “Yes” in response to the query to evaluate initialization cells.

```
SetDirectory[NotebookDirectory[]];
(* set directory where source files are located *)
Get["NTGUtilityFunctions.m"]; (* Load utilities package *)
```

Purpose

This is the 2nd in a series of notebooks in which I work through material and exercises in the magisterial new book *Modern Classical Physics* by Kip S. Thorne and Roger D. Blandford. If you are a physicist of any ilk, BUY THIS BOOK. You will learn from a close reading and from solving the exercises. See TB 13.4 Polytropes 01-01-18 for more discussion of this project.

Exercise 13.6 *Problem: Shapes of Stars in a Tidally Locked Binary System*

Consider two stars, with the same mass M orbiting each other in a circular orbit with diameter (separation between the stars' centers) a . Kepler's laws tell us that the stars' orbital angular velocity is $\Omega = \sqrt{2GM/a^3}$. Assume that each star's mass is concentrated near its center so that everywhere except near a star's center the gravitational potential, in an inertial frame, is $\Phi = -GM/r_1 - GM/r_2$ with r_1 and r_2 the distances of the observation point from the center of star 1 and star 2. Suppose that the two stars are “tidally locked”, i.e. tidal gravitational forces have driven them each to rotate with rotational angular velocity equal to the orbital angular velocity Ω . (The moon is tidally locked to the earth; that is why it always keeps the same face toward the earth.) Then in a reference frame that rotates with angular velocity Ω , each star's gas will be at rest, $\mathbf{v} = 0$.

- Write down the total potential $\Phi + \Phi_{\text{cen}}$ for this binary system in the rotating frame.
- Using Mathematica or Maple or some other computer software, plot the equipotentials $\Phi + \Phi_{\text{cen}} = (\text{constant})$ for this binary in its orbital plane, and use these equipotentials to describe the shapes that these stars will take if they expand to larger and larger radii (with a and M held fixed). You should obtain a sequence in which the stars, when compact, are well separated and nearly round; and as they grow, tidal gravity elongates them, ultimately into tear-drop shapes followed by merger into a single, highly distorted star. With further expansion there should come a point where the merged star starts flinging mass off into the surrounding space (a process not included in this hydrostatic analysis).

Analysis and solution

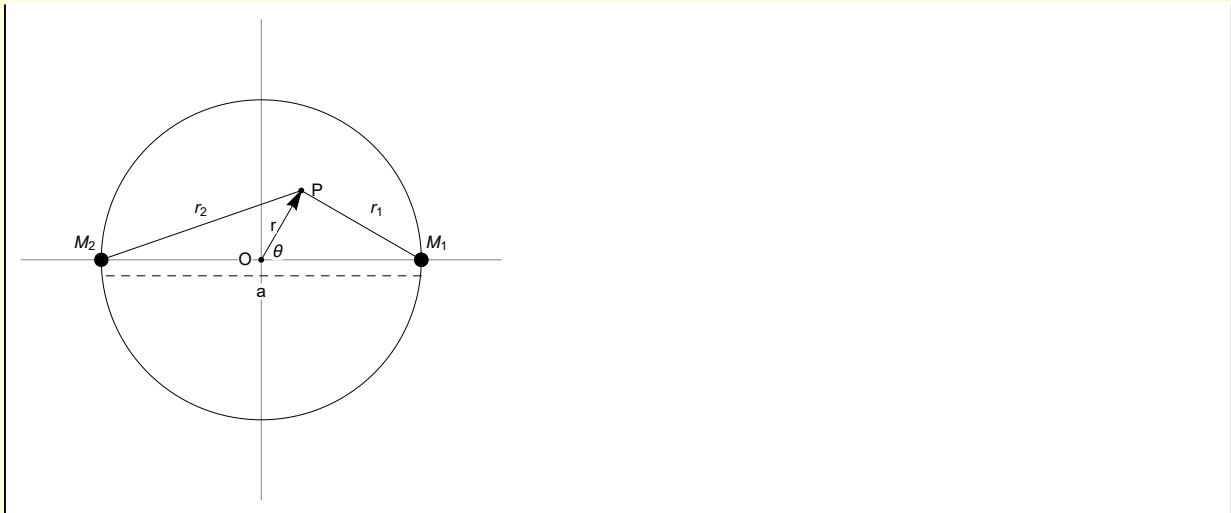


Figure 1 Geometry of the rotating binary star system

With equal masses, the dynamics are simplified. To obtain circular orbits, it is only necessary to equate the centrifugal force with the gravitational attraction.

$$w[1] = \frac{M v^2}{r} = \frac{G M^2}{(2r)^2}$$

$$\frac{M v^2}{r} = \frac{G M^2}{4 r^2}$$

where each star has mass M and move in a circle with radius r . The rotation period is

$$w[2] = T = \frac{2\pi r}{v} \quad /. \text{Solve}[w[1], v] \quad [[2, 1]]$$

$$T = \frac{4\pi r^{3/2}}{\sqrt{G} \sqrt{M}}$$

Squaring this expression yields Kepler's third law — the Law of Periods. The angular velocity is

$$w[3] = \Omega = \frac{2\pi}{T} \quad /. \text{Sol}[w[2], T]$$

$$\Omega = \frac{\sqrt{G} \sqrt{M}}{2 r^{3/2}}$$

TB expresses this in terms of $a = 2r$, the separation of the stars

$$w[4] = w[3] /. r \rightarrow a/2$$

$$\Omega = \frac{\sqrt{2} \sqrt{G} \sqrt{M}}{a^{3/2}}$$

If the mass of the stars is concentrated at their centers, the gravitational potential is

$$w[5] = \Phi_g = -\frac{GM_1}{r_1} - \frac{GM_2}{r_2}$$

$$\Phi_g = -\frac{GM_1}{r_1} - \frac{GM_2}{r_2}$$

The centrifugal potential is

$$\Phi_c = -\frac{1}{2} (\vec{r} \times \vec{\Omega})^2 = -\frac{1}{2} r^2 \Omega^2 (\mathbf{1}_r \times \mathbf{1}_z) \cdot (\mathbf{1}_r \times \mathbf{1}_z) = -\frac{1}{2} r^2 \Omega^2$$

$$w[6] = \Phi_c = -\frac{1}{2} r^2 \Omega^2$$

$$\Phi_c = -\frac{1}{2} r^2 \Omega^2$$

By the law of cosines

$$w[7] = r_1^2 = r^2 + (a/2)^2 - 2r(a/2)\cos[\theta]$$

$$r_1^2 = \frac{a^2}{4} + r^2 - ar\cos[\theta]$$

$$w[8] = r_2^2 = r^2 + (a/2)^2 - 2r(a/2)\cos[\pi - \theta]$$

$$r_2^2 = \frac{a^2}{4} + r^2 + ar\cos[\theta]$$

$$w[9] = \Phi_{\text{tot}} = \Phi_g + \Phi_c /. (w[5] // ER) /. (w[6] // ER)$$

$$\Phi_{\text{tot}} = -\frac{GM_1}{r_1} - \frac{GM_2}{r_2} - \frac{r^2 \Omega^2}{2}$$

$$w[10] = w[9] /. \text{Solve}[w[7], r_1][[2, 1]] /. \text{Solve}[w[8], r_2][[2, 1]]$$

$$\Phi_{\text{tot}} = -\frac{1}{2} r^2 \Omega^2 - \frac{2GM_1}{\sqrt{a^2 + 4r^2 - 4ar\cos[\theta]}} - \frac{2GM_2}{\sqrt{a^2 + 4r^2 + 4ar\cos[\theta]}}$$

```
w[11] = w[10] /. M1 -> M /. M2 -> M /. (w[4] // ER)
```

$$\Phi_{\text{tot}} = -\frac{GM r^2}{a^3} - \frac{2GM}{\sqrt{a^2 + 4r^2 - 4ar \cos[\theta]}} - \frac{2GM}{\sqrt{a^2 + 4r^2 + 4ar \cos[\theta]}}$$

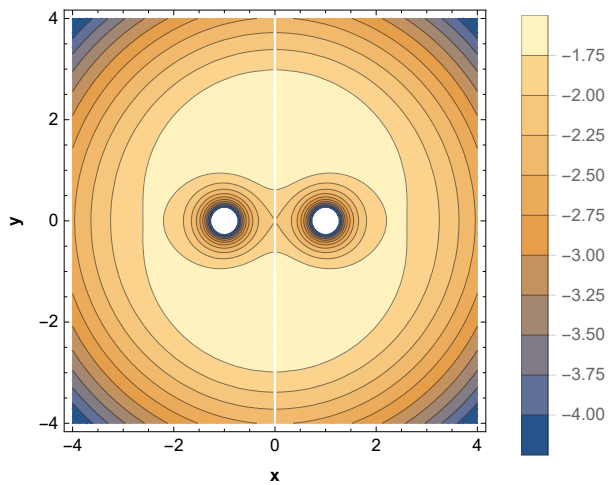
```
Clear[BinaryStarsLocked];
```

```
BinaryStarsLocked[r_, theta_, a_, G_, M_] :=
```

$$-\frac{GM r^2}{a^3} - \frac{2GM}{\sqrt{a^2 + 4r^2 - 4ar \cos[\theta]}} - \frac{2GM}{\sqrt{a^2 + 4r^2 + 4ar \cos[\theta]}}$$

```
Module[{M = 1, G = 1, a = 2, rMax = 4},
```

```
ContourPlot[BinaryStarsLocked[Sqrt[x^2 + y^2], ArcTan[y/x], a, G, M],
{x, -rMax, rMax}, {y, -rMax, rMax},
FrameLabel -> {St1["x"], St1["y"]}, Contours -> 10,
PlotLegends -> Automatic, ImageSize -> 250]
```



Equipotentials of tidally locked binary stars. Nominal parameters are $a = 2$, $G = 1$, $M = 1$.

For fixed a , the transition from circles (near the stars) to a tear drops (r of order $a/2$) to a single distorted shape ($r > a$) is apparent.

Visualization

```

Module[{M = 1, O = {0, 0}, a = 2, r = 0.5,  $\theta = \pi/3$ , xMax = 1.5, S1, S2, P,
  ptO, ptP, ptS1, ptS2, lineP, lineS1P, lineS2P, linea, orbit, txt $\theta$ , axes},
  axes = {Gray, Line[{{-xMax, 0}, {xMax, 0}}, Line[{{0, -xMax}, {0, xMax}}]};
  S1 = {a/2, 0.0}; S2 = {-a/2, 0.0};
  P = {r Cos[ $\theta$ ], r Sin[ $\theta$ ]};
  ptO = {Point[O], Text["O", O + {-0.1, 0}]};
  ptP = {Point[P], Text["P", P + {0.1, 0}]};
  ptS1 = {Point[S1], Text["M1", S1 + {0.1, 0.1}]};
  ptS2 = {Point[S2], Text["M2", S2 + {-0.1, 0.1}]};
  orbit = Circle[{0, 0}, a/2];
  linea = {Directive[Black, Dashed],
    Line[{S1 + {0, -0.1}, S2 + {0, -0.1}}, Text["a", {0, -0.2}]};
  lineP = {Arrow[{O, P}], Text["r",  $\frac{O + P}{2} + \{-0.05, 0\}$ ]};
  lineS1P = {Line[{S1, P}], Text["r1",  $\frac{S1 + P}{2} + \{0.1, 0.1\}$ ]};
  lineS2P = {Line[{S2, P}], Text["r2",  $\frac{S2 + P}{2} + \{0, 0.1\}$ ]};
  txt $\theta$  = Text[" $\theta$ ", {0.1, 0.05}];

  Graphics[{axes, txt $\theta$ , orbit, linea, {PointSize[0.03], Point[S1], ptS1, ptS2},
    ptO, ptP, lineP, lineS1P, lineS2P}, AspectRatio → 1,
    PlotRange → {{-xMax, xMax}, {-xMax, xMax}}, ImageSize → 250]

```

