TB 13.9 Crocco’s Theorem 01-19-18

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Initialization: Be sure the file *NTGUtilityFunctions.m* is in the same directory as that from which this notebook was loaded. Then execute the cell immediately below by mousing left on the cell bar to the right of that cell and then typing “shift” + “enter”. Respond “Yes” in response to the query to evaluate initialization cells.

```
SetDirectory[NotebookDirectory[]];
(* set directory where source files are located *)
Get["NTGUtilityFunctions.m"]; (* Load utilities package *)
```

Purpose

This is the 4th in a series of notebooks in which I work through material and exercises in the magisterial new book *Modern Classical Physics* by Kip S. Thorne and Roger D. Blandford. If you are a physicist of any ilk, BUY THIS BOOK. You will learn from a close reading and from solving the exercises.

Exercise 13.9 *Problem: Crocco’s Theorem*

(a) Consider steady flow of an ideal fluid. The Bernoulli function is conserved along streamlines. Show that the variation of B across streamlines is given by

$$\nabla B = T\nabla s + \mathbf{v} \times \boldsymbol{\omega} \,. \tag{13.64}$$

(b) As an example, consider the air in a tornado. In the tornado’s core, the velocity vanishes; and it also vanishes beyond the tornado’s outer edge. Use Crocco’s theorem to show that the pressure in the core is substantially different from that at the outer edge. Is it lower, or is it higher? How does this explain the ability of a tornado to make the walls of a house explode?

Analysis and solution

Part (a)

I start with a quick derivation of Crocco’s theorem. The Euler equation is

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\frac{\nabla P}{\rho} \tag{1}$$

Use the identity

$$\boldsymbol{v} \times \nabla \times \boldsymbol{v} = \frac{1}{2} \nabla v^2 - \boldsymbol{v} \cdot \nabla \boldsymbol{v} \tag{2}$$

to obtain

$$\frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{2} \nabla v^2 - \boldsymbol{v} \times \nabla \times \boldsymbol{v} = -\frac{\nabla P}{\rho} \tag{3}$$

Define the vorticity

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} \tag{4}$$

and

$$\frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{2} \nabla v^2 - \boldsymbol{v} \times \boldsymbol{\omega} = -\frac{\nabla P}{\rho} \tag{5}$$

Consider the representation of the second law of thermodynamics

$$T\,\mathrm{dS} = \mathrm{dh} - \frac{\mathrm{dP}}{\rho} \tag{6}$$

Along a stream line

$$T \nabla S = \nabla h - \frac{\nabla P}{\rho} \tag{7}$$

Using this to eliminate P

$$\frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{2} \nabla v^2 - \boldsymbol{v} \times \boldsymbol{\omega} = T \nabla S - \nabla h \tag{8}$$

Rewrite to emphasize the sources of entropy change and the result in Crocco’s theorem/equation.

$$T \nabla S = \frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{2} \nabla v^2 - \boldsymbol{v} \times \boldsymbol{\omega} + \nabla h \tag{9}$$

This equation links compressible fluid dynamics and thermodynamics. For stationary flow

$$T \nabla S = \frac{1}{2} \nabla v^2 - \boldsymbol{v} \times \boldsymbol{\omega} + \nabla h \tag{10}$$

The Bernoulli function is

$$B = \frac{v^2}{2} + h + \Phi \tag{11}$$

where, in this case, there is no gravitational field and Φ = 0.

Thus

$$\nabla B = \nabla\left(\frac{v^2}{2} + h\right) = T\,\nabla S + \boldsymbol{v} \times \boldsymbol{\omega} \tag{12}$$

Part (b)

If v is equal to zero both inside and outside the tornado

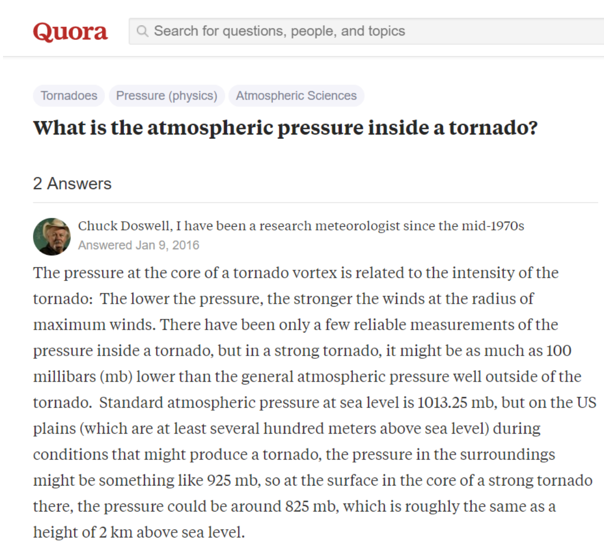
$$\nabla B = \nabla h = T\,\nabla S = \frac{\nabla P}{\rho}$$

Then, approximately,

$$T\,(S_{\text{outside}} - S_{\text{inside}}) = \frac{1}{\rho}\,(P_{\text{outside}} - P_{\text{inside}})$$

Since there is a large increase of entropy (disorder) from the inside to the outside of a tornado, the lhs is large positive and

$$P_{\text{outside}} >> \text{P}_{\text{inside}}$$



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Tornadoes Pressure (physics) Atmospheric Sciences

What is the atmospheric pressure inside a tornado?

2 Answers

Chuck Doswell, I have been a research meteorologist since the mid-1970s
Answered Jan 9, 2016

The pressure at the core of a tornado vortex is related to the intensity of the tornado: The lower the pressure, the stronger the winds at the radius of maximum winds. There have been only a few reliable measurements of the pressure inside a tornado, but in a strong tornado, it might be as much as 100 millibars (mb) lower than the general atmospheric pressure well outside of the tornado. Standard atmospheric pressure at sea level is 1013.25 mb, but on the US plains (which are at least several hundred meters above sea level) during conditions that might produce a tornado, the pressure in the surroundings might be something like 925 mb, so at the surface in the core of a strong tornado there, the pressure could be around 825 mb, which is roughly the same as a height of 2 km above sea level.

References

Detailed discussion of vorticity
file:///C:/Users/NTG/Downloads/9780387261409-c1%20(1).pdf

Another detailed discussion plus tornados
http://www.pmaweb.caltech.edu/Courses/ph136/yr2012/1214.1.K.pdf

http://www.astro.yale.edu/vdbosch/astro320_summary11.pdf