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# Kelvin Ship Waves 07-12-18

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**Initialization:** Be sure the file *NTGUtilityFunctions.m* is in the same directory as that from which this notebook was loaded. Then execute the cell immediately below by mousing left on the cell bar to the right of that cell and then typing “shift” + “enter”. Respond “Yes” in response to the query to evaluate initialization cells.

```
SetDirectory[NotebookDirectory[]];  
(* set directory where source files are located *)  
Get["NTGUtilityFunctions.m"]; (* Load utilities package *)
```

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## I Introduction

The distinctive wave/wake pattern behind a moving ship is a familiar one and, remarkably, the pattern is similar whether the moving object is a ship at sea or a duck in a pond. The detailed physical-mathematical explanation of this phenomenon by Lord Kelvin, was a triumph of 19th century physics.

[https://en.wikipedia.org/wiki/William\\_Thomson,\\_1st\\_Baron\\_Kelvin](https://en.wikipedia.org/wiki/William_Thomson,_1st_Baron_Kelvin)

This seminal work by Lord Kelvin stimulated much further research and general interest. A Google search just performed on “Kelvin ship waves” generated 453000 hits.



Ship waves interest me too. Below, I use Mathematica to work through some classical analysis related to these waves. Since the calculations are complicated and involve several approximations, the detailed step-by-step sequence of results that Mathematica makes possible may be of interest to students and nonspecialist researchers.

I follow a procedure similar to Kelvin. Kelvin calculated the linear response of water to a point disturbance in deep water (imagine the ripples generated from a pebble dropped in water). He modeled a ship as a moving point disturbance. The cumulative response observed behind a moving ship is the

integral of the disturbances along the path of the ship. In general, waves will propagate radially outward from each point of disturbance with different wave lengths and phase velocities. A key point in the analysis is that an observer on the ship sees a wave pattern that is stationary in time.

The seminal document is *On the waves produced by a single impulse in water of any depth, or in a dispersive medium*. Proceedings of the Royal Society of London, Ser. A. Vol. 42, 1887, pp. 80-85. Lord Kelvin.

The history of the study of ship waves is complicated. I won't attempt a review. I will generally follow the treatment given in Chapter 8 of *Water Waves: The Mathematical Theory with Applications*, J. J. Stoker, 1957. Lord Kelvin is credited with inventing the method of stationary phase while working on this problem. In fact, in the course of the analysis, the method of stationary phase is invoked three times!

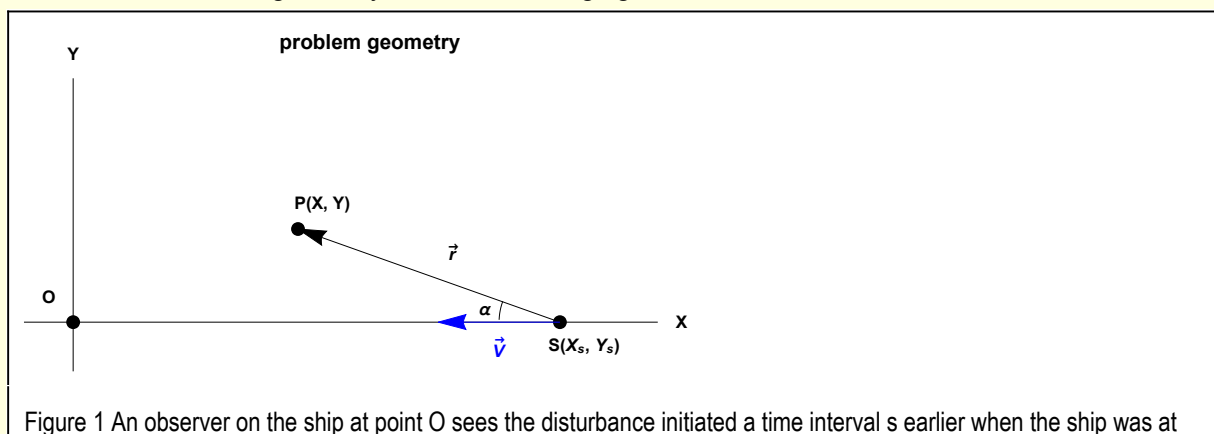
The importance of this problem for applied mathematics is underscored by the fact that the cover picture of *The Princeton Companion to Applied Mathematics* is an image of a Kelvin ship wave.

In order to get quickly to the heart of the subject, I will start with a particular approximate formula for the linear response of water to a point disturbance in deep water. Assuming that a ship can be modeled as a moving version of this point disturbance enables one to deduce the characteristic wave features behind the ship. Insight is also gained into the detailed physics associated with the generation of the wave pattern.

In a second section, I will derive the particular formula for the linear response from a more fundamental result. This is a longish calculation involving a double integral that is both analytically and numerically difficult. Along the way I will provide various visualizations that clarify aspects of the calculation.

Overall this is a calculation that, if you know where you are going, can be carried out with pen and paper in 20 or 30 pages of calculation. However, it is easy to go astray. As I do with the notebooks on this website I try to demonstrate that all of the substantive calculations can be carried out with Mathematica.

I establish notation and geometry with the following figure.



S.

The ship moves along the X-axis in the negative direction at constant speed  $V$ . The analysis is performed in the reference frame of the moving ship. The ship's position is parameterized by the positive quantity  $s$ . The point S denotes the position of the ship  $\{X_s, Y_s\} = \{X_s(s) = \{V s, 0\}$ . The point O denotes the position of the ship when  $s = 0$ . Thus, the parameter  $s$  corresponds to the positive interval of time required for the ship to move from S to O, while moving at constant speed  $V$ . The point  $P(X, Y)$  is a point being observed from O, when  $s = 0$ . The quantity  $\vec{r}$  is a vector from S to P and  $\alpha$  is the angle between the velocity vector of the ship and  $\vec{r}$ .

The quantity of interest is the cumulative disturbance at  $P(X, Y)$

$$\delta z(P) = \int_0^{s_{\max}} ds \delta z(\vec{r}(s), s) \quad (1)$$

where  $\delta z$  is some infinitesimal change in the level of water and  $s_{\max}$  is some arbitrary large interval during which the ship has been moving and generating waves. The basic physical model for ship-generated waves is that of the waves generated from a point source. Another way of imagining the model is — an experimenter sits on a platform above still deep water that is flowing past with speed  $V$ . The experimenter continuously drops pebbles into the water and observes the stationary wave pattern that forms in the downstream flow of water.

The general linear response of water near the surface to an instantaneous point source is

$$\delta z_{\text{initial value problem}}(r, t) = - \frac{1}{2 \pi \rho \sqrt{g}} \int_0^{\infty} dk k^{3/2} J_0(k r) \sin(\omega_k t) \quad (2)$$

$$\omega_k = \sqrt{k g}$$

where  $g$  is the gravitational acceleration,  $\rho$  is the density of water, and  $J_0(x)$  is a Bessel function of order 0. The wave frequency  $\omega_k$  is appropriate for deep water (the derivation of the dispersion equation for water waves is presented in another notebook). The integrand of this expression becomes highly oscillatory as  $k$  increases (product of two different oscillating functions each depending on  $k$ ) and its valuation is challenging even using computational methods. A classic reference for equation (2) is *Hydrodynamics*, Lamb, 1932 (still available as a Dover book).

In Section 2 below, the following approximate form for equation (2) will be calculated.

$$\delta z(r, t) \cong \frac{g t^3}{8 \sqrt{2} \pi \rho r^4} \sin\left(\frac{g t^2}{4 r}\right) \quad (3)$$

The objective of Section 1 is to evaluate and analyze the integral (1) with argument (3)

$$\delta z(P) = \kappa \int_0^{s_{\max}} ds \frac{s^3}{r^4} \exp\left(i \frac{g s^2}{4 r}\right) \equiv \kappa \int_0^{s_{\max}} ds \Psi(r, s) e^{i \Phi(r, s)} \quad (4)$$

where  $\kappa$  is a constant and it is understood that it is the imaginary part of  $\delta z(r, t)$  that is physically relevant.

## 2 Calculation of $\delta z(r, t)$

The integrand of (4) is highly oscillatory and is calculated using the method of stationary phase. The calculation not only produces an approximate value for  $\delta z(r, t)$ , but also provides insight about the wave pattern. I start by defining some functions for future convenience

```
Clear[Ψ, Φ];
Ψ[r_, s_] :=  $\frac{s^3}{r^4}$ ;
Φ[r_, s_] :=  $\frac{g s^2}{4 r}$ ;
```

The constant phase condition,  $d\Phi/dr = 0$  results in a differential equation

```
w2[1] = Sol[D[Φ[r[s], s], s] == 0, r'[s]] // RE
```

$$r'[s] == \frac{2 r[s]}{s}$$

```
w2["st phase ode"] = w2[1]
```

$$r'[s] == \frac{2 r[s]}{s}$$

having the solution

```
w2[2] = DSolve[{w2[1], r[s0] == r0}, r[s], s][[1, 1]]
```

$$r[s] \rightarrow \frac{r_0 s^2}{s_0^2}$$

At this point, there is no natural scale length so I show  $\Phi(r, s)$  and the contours of constant  $\Phi$  in nominal physical coordinates for  $g = 9.8 \text{ m/sec}^2$

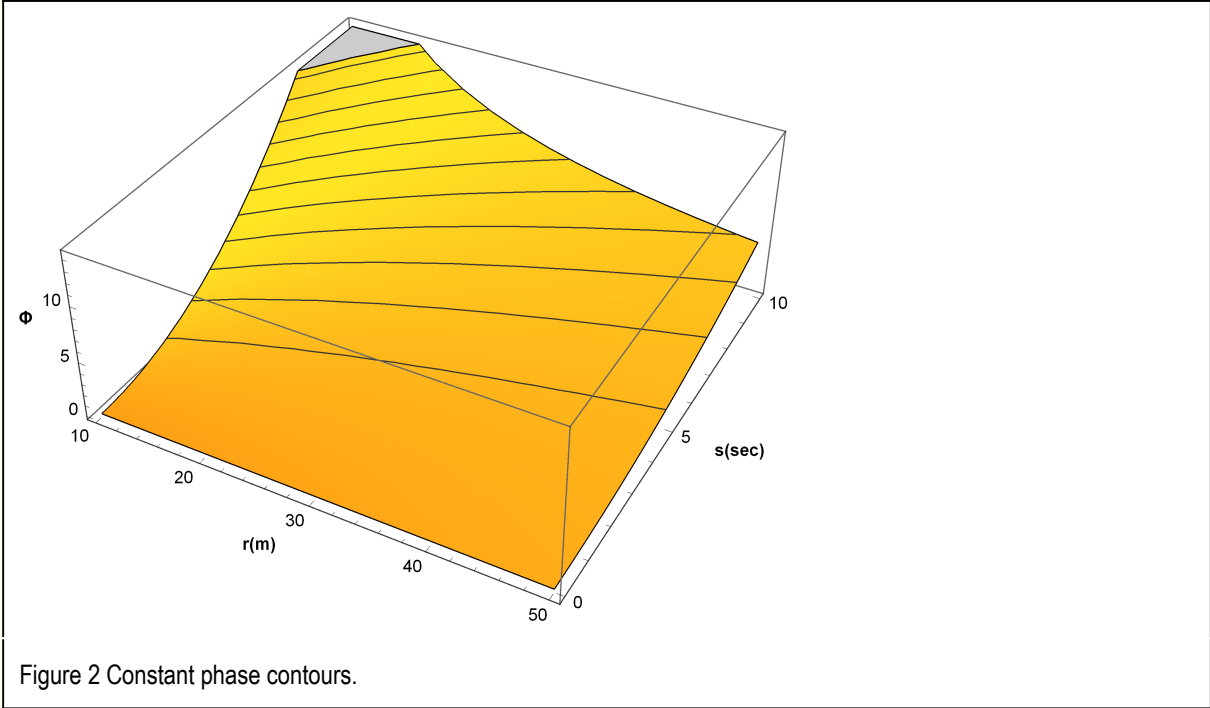


Figure 2 Constant phase contours.

What does the stationary phase requirement imply in the context of the problem geometry? Note

$$w2[3] = r[s] = \sqrt{(XS[s] - X)^2 + YS[s]^2}$$

$$r[s] = \sqrt{(-X + XS[s])^2 + YS[s]^2}$$

For the assumed ship motion

$$w2[4] =$$

$$D[w2[3], s] /. XS'[s] \to V /. YS'[s] \to 0 /. 1/\sqrt{(-X + XS[s])^2 + YS[s]^2} \to 1/r[s]$$

$$r'[s] = \frac{V(-X + XS[s])}{r[s]}$$

Recall that the ship is moving in the -X direction and so  $xS'(t) = -V$ . However, in this calculation the positive interval of time  $s$  is being used as the "time-variable" and  $xS'(s) = +V$ . That is, the ship's position on the x-axis increases as  $s$  increases.

This expression can be simplified

$$w2[5] = w2[4] /. Sol[(XS[s] - X)/r[s] == Cos[\alpha[s]], XS[s]]$$

$$r'[s] = V Cos[\alpha[s]]$$

$$w2["st phase r ode"] = w2[5]$$

$$r'[s] == V \text{Cos}[\alpha[s]]$$

For future use,  $\alpha'(s)$  will be needed. From Figure 1 and some trigonometry

$$w2[6] = \text{Tan}[\alpha[s]] == \frac{Y}{(XS[s] - X)} /. XS[s] \rightarrow sV$$

$$\text{Tan}[\alpha[s]] == \frac{Y}{sV - X}$$

$$w2[7] = \text{MapEqn}[D[\#, s] \&, w2[6]] /. \text{Sol}[w2[6], Y]$$

$$\text{Sec}[\alpha[s]]^2 \alpha'[s] == -\frac{V \text{Tan}[\alpha[s]]}{sV - X}$$

$$w2[8] = \text{Sol}[w2[7], \alpha'[s]] /. sV - X \rightarrow r \text{Cos}[\alpha[s]] /. r \rightarrow r[s]$$

$$\alpha'[s] \rightarrow -\frac{V \text{Sin}[\alpha[s]]}{r[s]}$$

$$w2["\text{st phase } \alpha \text{ ode}"] = w2[8]$$

$$\alpha'[s] \rightarrow -\frac{V \text{Sin}[\alpha[s]]}{r[s]}$$

In terms of quantities that describe the ship motion, the stationary phase conditions implies

$$w2[9] = w2[1] /. (w2[5] // ER)$$

$$V \text{Cos}[\alpha[s]] == \frac{2r[s]}{s}$$

or

$$w2[10] = \text{Sol}[w2[9], r[s]] // RE$$

$$r[s] == \frac{1}{2} s V \text{Cos}[\alpha[s]]$$

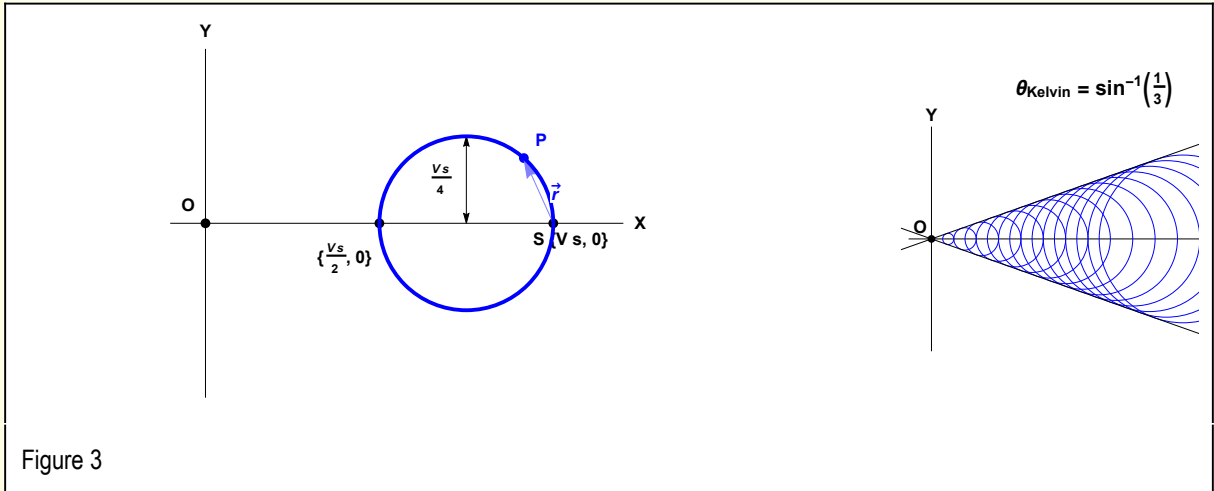
$$w2["\text{circle of influence}"] = w2[10]$$

$$r[s] == \frac{1}{2} s V \text{Cos}[\alpha[s]]$$

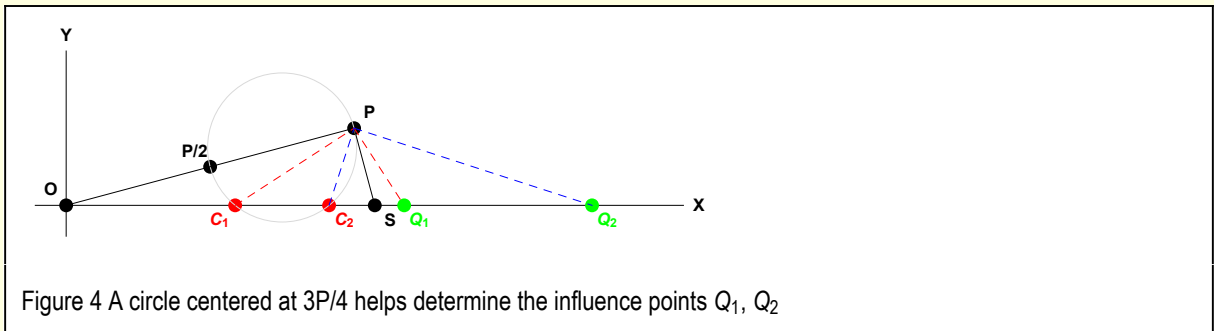
This radius defines a “circle of influence.” In the figure below, the points of the blue circle are the places where the stationary phase condition is satisfied, and where an observable disturbance is expected. Elsewhere, there are mixtures of oscillating waveforms that cancel each other.



The diameter of the circle is  $\frac{Vs}{2}$  and the center of the circle is located at  $\frac{3Vs}{4}$ . The right panel shows a sequence of circles of influence as  $s$  decreases and the ship approaches  $O$ . It is clear that the observable disturbances are confined to a wedge behind the ship. Consider a line tangent to these circles: the half-angle between that line and the  $x$ -Axis is the Kelvin angle  $\theta_{\text{Kelvin}} = \sin^{-1}(1/3)$ . Famously, it is independent of the speed of the ship.



I follow Stoker who used a geometric figure developed by Lamb to elucidate more physical insight into the nature of the waves caused by the ship.



As before, the points  $O$  and  $S$  denote the position of the ship when  $s = 0$  and  $s = s$ . The point  $P$  has been chosen so that  $OP$  is orthogonal to  $PS$ . The gray circle is constructed and its intersections with  $x$ -axis is calculated. There are either two points of intersection  $C_1$  and  $C_2$ , or none at all. The points  $Q_1$  and  $Q_2$  are constructed so that  $C_1P \perp Q_1P$  and  $C_2P \perp Q_2P$ . These points are specific “influence points” that give rise to different types of waves.

The next step is to calculate the phase fronts in the frame of Figure 1: to be specific,  $X = X(\alpha)$ ,  $Y = Y(\alpha)$ . From Figure 1

$$\begin{aligned} \mathbf{w}_2[11] &= \{XS - X = r \cos[\alpha], Y = r \sin[\alpha]\} \\ \{-X + XS &= r \cos[\alpha], Y = r \sin[\alpha]\} \end{aligned}$$

The ship is located at  $Vs$



$$w2[12] = w2[11] /. XS \to V s$$

$$\{s V - X == r \cos[\alpha], Y == r \sin[\alpha]\}$$

Choose some reference value of the phase  $\phi_0$

$$w2[13] = \phi_0 == \Phi[r, s]$$

$$\phi_0 == \frac{g s^2}{4 r}$$

and recall the circle of influence equations

$$w2[14] = w2["circle of influence"] /. r[s] \to r /. \alpha[s] \to \alpha$$

$$r == \frac{1}{2} s V \cos[\alpha]$$

Solve the equations

$$\{w2[12][[1]], w2[12][[2]], w2[13], w2[14]\}$$

$$\{s V - X == r \cos[\alpha], Y == r \sin[\alpha], \phi_0 == \frac{g s^2}{4 r}, r == \frac{1}{2} s V \cos[\alpha]\}$$

$$w2[13] = \text{Solve}[\{w2[12][[1]], w2[12][[2]], w2[13], w2[14]\}, \{X, Y, r, s\}][[1]]$$

$$\left\{ X \rightarrow \frac{2 V^2 \phi_0 \cos[\alpha] - V^2 \phi_0 \cos[\alpha]^3}{g}, \right. \\ \left. Y \rightarrow \frac{V^2 \phi_0 \cos[\alpha]^2 \sin[\alpha]}{g}, r \rightarrow \frac{V^2 \phi_0 \cos[\alpha]^2}{g}, s \rightarrow \frac{2 V \phi_0 \cos[\alpha]}{g} \right\}$$

Notice that the following quantity has the dimensions of length

$$\text{def}[a_0] = a_0 == V^2 \phi_0 / g$$

$$a_0 == \frac{V^2 \phi_0}{g}$$

$$w2[14] = w2[13] /. \text{Sol}[\text{def}[a_0], g] // \text{Simplify}$$

$$\left\{ X \rightarrow -a_0 \cos[\alpha] (-2 + \cos[\alpha]^2), Y \rightarrow a_0 \cos[\alpha]^2 \sin[\alpha], r \rightarrow a_0 \cos[\alpha]^2, s \rightarrow \frac{2 a_0 \cos[\alpha]}{V} \right\}$$

The specific quantities of interest are  $X(\alpha)$  and  $Y(\alpha)$

```
w2[15] = w2[14][[1 ;; 2]]
```

```
{X → -a0 Cos[α] (-2 + Cos[α]2), Y → a0 Cos[α]2 Sin[α]}
```

```
w2["phase fronts"] = w2[15]
```

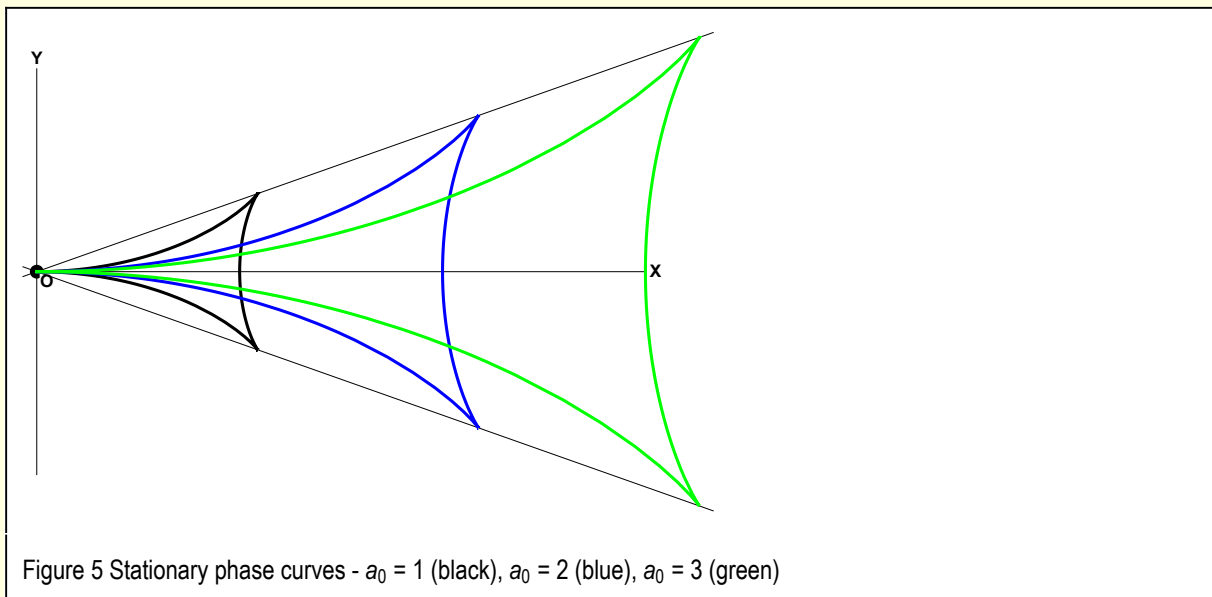
```
{X → -a0 Cos[α] (-2 + Cos[α]2), Y → a0 Cos[α]2 Sin[α]}
```

Define a parametric representation for the stationary phase curve characterized by phase value  $\phi_0$  (or length  $a_0$ ).

```
Clear[ϕCurve];
```

```
ϕCurve[α_, a0_] := {-a0 Cos[α] (-2 + Cos[α]2), a0 Cos[α]2 Sin[α]}
```

I plot some representative curves along which the phase is stationary



The value of  $\alpha$  for which  $Y(\alpha)$  is maximum (the cusp) can be calculated

```
w2[16] = Solve[D[ϕCurve[α, a][2], α] == 0, α]
{{α → ConditionalExpression[-π/2 + 2π C[1], C[1] ∈ Integers]},
 {α → ConditionalExpression[π/2 + 2π C[1], C[1] ∈ Integers]},
 {α → ConditionalExpression[-ArcTan[1/√2] + 2π C[1], C[1] ∈ Integers]},
 {α → ConditionalExpression[π - ArcTan[1/√2] + 2π C[1], C[1] ∈ Integers]},
 {α → ConditionalExpression[ArcTan[1/√2] + 2π C[1], C[1] ∈ Integers]},
 {α → ConditionalExpression[-π + ArcTan[1/√2] + 2π C[1], C[1] ∈ Integers]}}
```

Specifically, for positive Y

```
w2[17] = w2[16][5, 1] /. C[1] → 0 /. α → αcusp
```

$$\alpha_{\text{cusp}} \rightarrow \text{ArcTan}\left[\frac{1}{\sqrt{2}}\right]$$

```
w2["αCusp"] = w2[17]
```

$$\alpha_{\text{cusp}} \rightarrow \text{ArcTan}\left[\frac{1}{\sqrt{2}}\right]$$

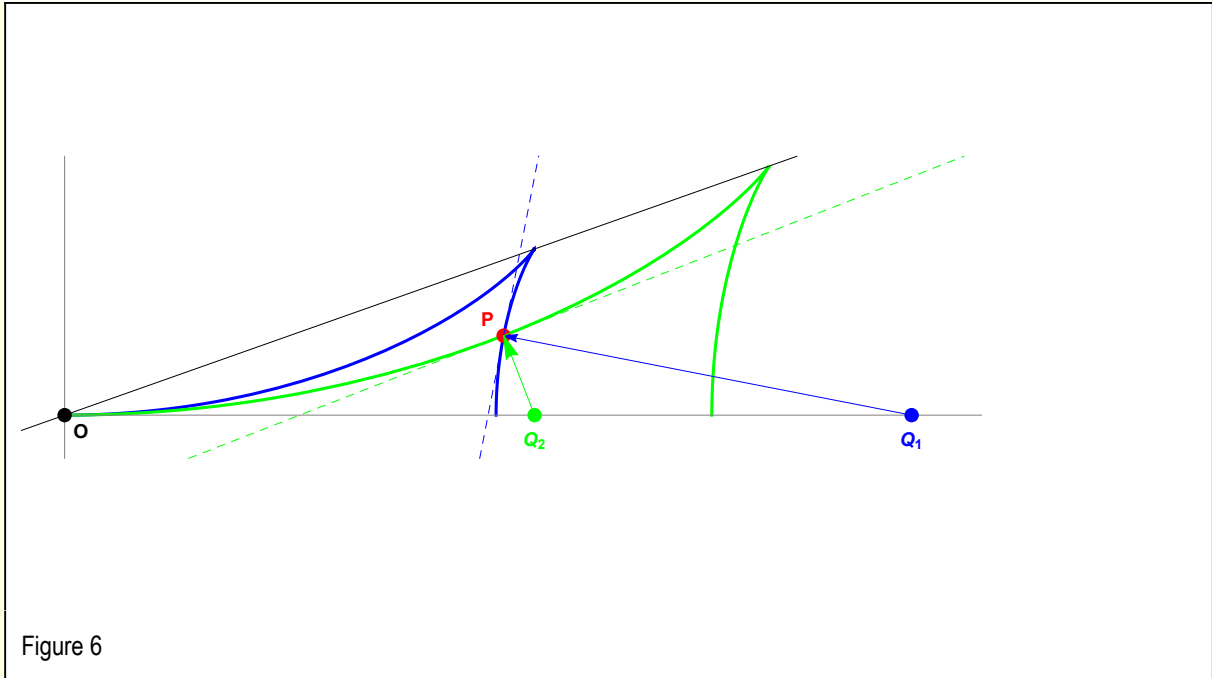
For  $\alpha < \alpha_{\text{cusp}} = 0.615$ , Y is increasing from 0 to its maximum value. For  $\alpha > \alpha_{\text{cusp}}$  Y is decreasing back to 0.

Consider the point on Figure 4 where the blue curve intersects the green curve. It can be determined numerically.

```
w2[18] = With[{a2 = 2, a3 = 3}, FindRoot[{ϕCurve[α1, a2][1] == ϕCurve[α2, a3][1],
 ϕCurve[α1, a2][2] == ϕCurve[α2, a3][2]}, {{α1, 0.615 - 0.2}, {α2, 0.615 + 0.2}}]]
{α1 → 0.192424, α2 → 1.19923}
```

where I have assisted the root finder by guessing  $\alpha_1 < \alpha_{\text{cusp}}$  and  $\alpha_2 > \alpha_{\text{cusp}}$ .

Next I construct the tangent to the green curve at the intersection point (the dashed green line). The normal to green curve at the intersection point intersects the ship path at  $Q_1$ . Similarly, the intersection for the normal to the blue curve is  $Q_2$ .



The points  $Q_1$  and  $Q_2$  are the influence points for the disturbance at  $P$  — that is, the points along the ship's path that strongly influence the disturbance at point  $P$ . There is a contribution from a circular wave packet emanating from  $Q_1$  and propagated primarily in the minus  $X$ -direction, forming an acute angle with the ship's motion. This is called the divergent wave. The second contribution propagates outward from  $Q_2$  and is called the transverse wave. Recall that the observable part of a wave packet is where the energy is concentrated, and that the energy concentration propagates with the group velocity. Clearly, the group velocity of the wave packet originating from  $Q_1$  is larger than the group velocity of the wave packet originating from  $Q_2$ . In the photographs above, both of these wave patterns can be observed.

Returning to equation (4) for the linear response

$$\begin{aligned} \delta z(r, t) &= \kappa \int_0^{S_{\max}} ds \frac{s^3}{r^4} \exp\left(i \frac{g s^2}{4 r}\right) \equiv \kappa \int_0^{S_{\max}} ds \Psi(r, s) e^{i \Phi(r, s)} \\ &\cong \kappa \Psi(r_{\text{in}}, s_{\text{in}}) \sqrt{\frac{2 \pi}{|\Phi''(r_{\text{in}}, s_{\text{in}})|}} e^{i \Phi(r_{\text{in}}, s_{\text{in}})} e^{i \text{sign}(\Phi''(r_{\text{in}}, s_{\text{in}})) \pi/4} \end{aligned} \tag{5}$$

where the second line is the result of applying the stationary phase approximation. The formula is derived in Appendix C.

The observable response at point  $P$  is the consequence of waves emanating from  $Q_1$  and  $Q_2$ . For the explicit stationary phase evaluation, it is more convenient to use  $r$  and  $\alpha$  as the variables. Thus

$$\delta z(r, t) \cong \kappa \Psi(r_2, \alpha_2) \sqrt{\frac{2 \pi}{|\Phi''(r_2, \alpha_2)|}} e^{i \Phi(r_2, \alpha_2)} e^{i \text{sign}(\Phi''(r_2, \alpha_2)) \pi/4} +$$

$$\kappa \Psi(r_3, \alpha_3) \sqrt{\frac{2\pi}{|\Phi''(r_3, \alpha_3)|}} e^{i\Phi(r_3, \alpha_3)} e^{i\text{sign}(\Phi''(r_3, \alpha_3))\pi/4}$$

Determining the explicit forms for the components of this equation requires a rather elaborate calculation performed in Appendix B.

$$\left\{ \hat{\Phi}[r[sP], sP] = \frac{g s \text{Sec}[\alpha[s]]}{2V}, \right. \\ \hat{\Phi}^{(\theta,2)}[r[sP], sP] = \frac{g}{2r[s]} - \frac{g \text{Tan}[\alpha[s]]^2}{r[s]}, \text{sign}[\hat{\Phi}^{(\theta,2)}[r[sP], sP]] = (\alpha[s] < \alpha_{\text{Cusp}}), \\ \left. \text{sign}[\hat{\Phi}^{(\theta,2)}[r[sP], sP]] = -(\alpha[s] > \alpha_{\text{Cusp}}) \right\} // \text{ColumnForm}$$

$$\hat{\Phi}[r[sP], sP] = \frac{g s \text{Sec}[\alpha[s]]}{2V} \\ \hat{\Phi}^{(\theta,2)}[r[sP], sP] = \frac{g}{2r[s]} - \frac{g \text{Tan}[\alpha[s]]^2}{r[s]} \\ \text{sign}[\hat{\Phi}^{(\theta,2)}[r[sP], sP]] = (\alpha[s] < \alpha_{\text{Cusp}}) \\ \text{sign}[\hat{\Phi}^{(\theta,2)}[r[sP], sP]] = -(\alpha[s] > \alpha_{\text{Cusp}})$$

As illustrated in Appendix C, the method of stationary phase usually involves a formal large parameter. In the calculations above, there is no such formal large parameter. However, Stoker introduces some dimensionless variables

$$w2[19] = \Phi[r, s] /. r \rightarrow \mathcal{R} R_0 /. s \rightarrow \mathcal{S} S_0$$

$$\frac{g S_0^2 \mathcal{S}^2}{4 R_0 \mathcal{R}}$$

where  $\mathcal{R}$  and  $\mathcal{S}$  are dimensionless forms for space and time variation. The only natural scale length in the problem is  $R_0$ , namely the distance from O to P. There is no natural time scale but  $S_0$  can be chosen such that  $R_0/S_0 = V$ , the speed of the ship. Then

$$w2[20] = w2[19] /. S_0 \rightarrow R_0/V$$

$$\frac{g R_0 \mathcal{S}^2}{4 V^2 \mathcal{R}}$$

Defining the dimensionless parameter  $\mathcal{K}$

$$\text{def}[\mathcal{K}] = \mathcal{K} = \frac{g R_0}{4 V^2}$$

$$\mathcal{K} = \frac{g R_0}{4 V^2}$$

$$w2[21] = \Phi == w2[20] /. Sol[def[\mathcal{K}], g]$$

$$\Phi == \frac{\mathcal{K} S^2}{\mathcal{R}}$$

and it is clear that  $\mathcal{K} \gg 1$  is required for the stationary phase approximation to be accurate. This corresponds to an observation point P far from the ship. When looking at the wake close to a moving ship, the waves, of course, are not stationary and the physics is much more complicated. The shape of the ship is important, the method of propulsion is important, the existing distortion of the water is important, and nonlinearities abound.

## Appendix A: Point disturbance in deep water (k integral)

In cylindrical coordinates, the linear response to a point disturbance at the surface of water, e.g. dropping a pebble in a pond, is

Linear response to a point disturbance at the surface of a fluid (Stoker 6.4.23)

$$wA1[1] = \delta z[r, t] == -\frac{1}{2\pi\rho\sqrt{g}} \text{Intk}[k^{3/2} J_0[kr] \text{Sin}[\sqrt{gk} t]]$$

$$\delta z[r, t] == -\frac{\text{Intk}[k^{3/2} J_0[kr] \text{Sin}[\sqrt{gk} t]]}{2\sqrt{g}\pi\rho}$$

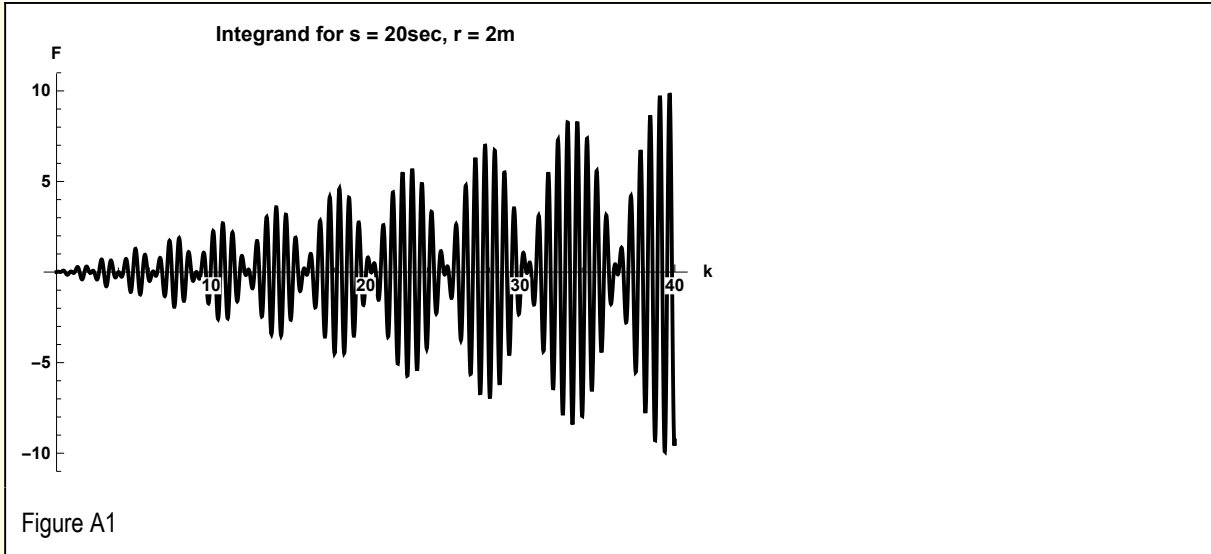
where  $\delta z$  is the surface elevation,  $g$  is gravitational acceleration,  $\rho$  is density. This result is derived in many places, e.g. Stoker *Water Waves*. Basically, it follows from solving the initial value problem for the wave equation in cylindrical coordinates for a delta function source in space and time.

The term `Intk` is a shorthand notation for an integral

$$\text{Intk}[f(k)] \equiv \int_0^\infty f(k) dk$$

I use `Intk` instead of `Integrate` because I want to perform manipulations without triggering the actual process of integration. I prefer this simple notational device to Mathematica functions such as `Hold` or `Inactivate`.

For parameters of physical interest, the integrand becomes highly oscillatory. I illustrate for some nominal physical parameters.



This integral is evaluated using the method of stationary phase. In fact, Lord Kelvin (William Thompson) developed that method of approximation while attacking this problem.

[https://en.wikipedia.org/wiki/Stationary\\_phase\\_approximation](https://en.wikipedia.org/wiki/Stationary_phase_approximation)

The classical form of integrals for which this approximation is appropriate is

$$I(k) = \int dx g(x) e^{i p f(x)}$$

The idea is that the integrand is highly oscillatory except when the phase  $f(x)$  is relatively “stationary”. The point of stationary phase is the value of  $x$  for which  $f'(x) = 0$ , and the integral can be approximated by restricting the range of integration to the region near the point of stationary phase. The accuracy increases as  $p$  becomes large and the region of stationary phase is more sharply localized

I perform a series of transformation that cast  $wA1[1]$  into the classical form. The first step is to express the Bessel function in terms of an integral over trigonometric functions

$$wA1[2] = wA1[1] / J_0[kr] \rightarrow \frac{2}{\pi} \text{Int}_{\beta} [\text{Cos}[kr \cos[\beta]]]$$

$$\delta z[r, t] = -\frac{1}{2\sqrt{g}\pi\rho} \text{Int}_k \left[ \frac{1}{\pi} 2k^{3/2} \text{Int}_{\beta} [\text{Cos}[kr \cos[\beta]]] \text{Sin}[\sqrt{gk}t] \right]$$

where

$$\text{Int}_{\beta}[f(\beta)] \equiv \int_0^{\pi/2} f(\beta) d\beta$$

Here  $\text{Cos}[\beta] = \cos[\beta]$  has been introduced temporarily to facilitate future manipulations

Change the order of integration



```
wA1[3] = wA1[2] /. a_Intk[b_Intβ[c_]] /. FreeQ[b, β] → a Intβ[Intk[b c]]
```

$$\delta z[r, t] == -\frac{1}{2\sqrt{g}\pi\rho} \text{Int}\beta\left[\text{Intk}\left[\frac{1}{\pi} 2 k^{3/2} \cos[k r \cos[\beta]] \sin[\sqrt{gk} t]\right]\right]$$

```
wA1[4] = MapAll[TrigToExp, wA1[3]] // ExpandAll
```

$$\delta z[r, t] == -\frac{1}{2\sqrt{g}\pi\rho} \text{Int}\beta\left[\text{Intk}\left[\frac{i e^{-i\sqrt{gk} t - i k r \cos[\beta]} k^{3/2}}{2\pi} - \frac{i e^{i\sqrt{gk} t - i k r \cos[\beta]} k^{3/2}}{2\pi} + \frac{i e^{-i\sqrt{gk} t + i k r \cos[\beta]} k^{3/2}}{2\pi} - \frac{i e^{i\sqrt{gk} t + i k r \cos[\beta]} k^{3/2}}{2\pi}\right]\right]$$

I need to manipulate the exponential terms. While the following works, there is probably a more elegant procedure.

```
wA1[5] = wA1[4] /. -i√gk t - i k r cos[β] → -I argP[k, β] /.  
i√gk t + i k r cos[β] → I argP[k, β] /.  
-i√gk t + i k r cos[β] → -I argM[k, β] /. i√gk t - i k r cos[β] → I argM[k, β]
```

$$\delta z[r, t] == -\frac{1}{2\sqrt{g}\pi\rho} \text{Int}\beta\left[\text{Intk}\left[\frac{i e^{-i \text{argM}[k, \beta]} k^{3/2}}{2\pi} - \frac{i e^{i \text{argM}[k, \beta]} k^{3/2}}{2\pi} + \frac{i e^{-i \text{argP}[k, \beta]} k^{3/2}}{2\pi} - \frac{i e^{i \text{argP}[k, \beta]} k^{3/2}}{2\pi}\right]\right]$$

Note

```
Exp[I arg] - Exp[-I arg] == 2 I Im[Exp[I arg]] // ComplexExpand
```

```
True
```

So

```
wA1[6] =  
wA1[5] /. Exp[-I argP[k, β]] → Exp[I argP[k, β]] - 2 I Im[Exp[I argP[k, β]]] /.  
Exp[-I argM[k, β]] → Exp[I argM[k, β]] - 2 I Im[Exp[I argM[k, β]]] // ExpandAll
```

$$\delta z[r, t] == -\frac{1}{2\sqrt{g}\pi\rho} \text{Int}\beta\left[\text{Intk}\left[\frac{k^{3/2} \text{Im}[e^{i \text{argM}[k, \beta]}]}{\pi} + \frac{k^{3/2} \text{Im}[e^{i \text{argP}[k, \beta]}]}{\pi}\right]\right]$$

To lessen the notational burden I shall henceforth that only the imaginary part of the rhs should be calculated

**wA1[7] = wA1[6] /. Im[a\_] -> a**

$$\delta z[r, t] == -\frac{1}{2\sqrt{g}\pi\rho}\text{Int}\beta\left[\text{Intk}\left[\frac{e^{i\text{argM}[k,\beta]}k^{3/2}}{\pi} + \frac{e^{i\text{argP}[k,\beta]}k^{3/2}}{\pi}\right]\right]$$

I introduce some simplifying operations for the integral operators. It is tempting to define such evaluation rules at the beginning of the calculation - but they evaluate immediately and interfere with some replacement rules.

```
Clear[Intk, Intβ];
Intk[a_b_] /; FreeQ[a, k] := a Intk[b];
Intβ[a_b_] /; FreeQ[a, β] := a Intβ[b];
Intk[a_ + b_] := Intk[a] + Intk[b];
Intβ[a_ + b_] := Intβ[a] + Intβ[b];
```

Back to the original notation. Also, since trigonometric manipulations are complete, the artificial form  $\cos[\beta]$  can be removed

**wA1[8] =**  
**wA1[7] /. argP[k, β] ->  $\sqrt{gk}t + kr\cos[\beta]$  /. argM[k, β] ->  $\sqrt{gk}t - kr\cos[\beta]$  /.  
 cos[a\_] -> Cos[a] // Expand**

$\delta z[r, t] ==$

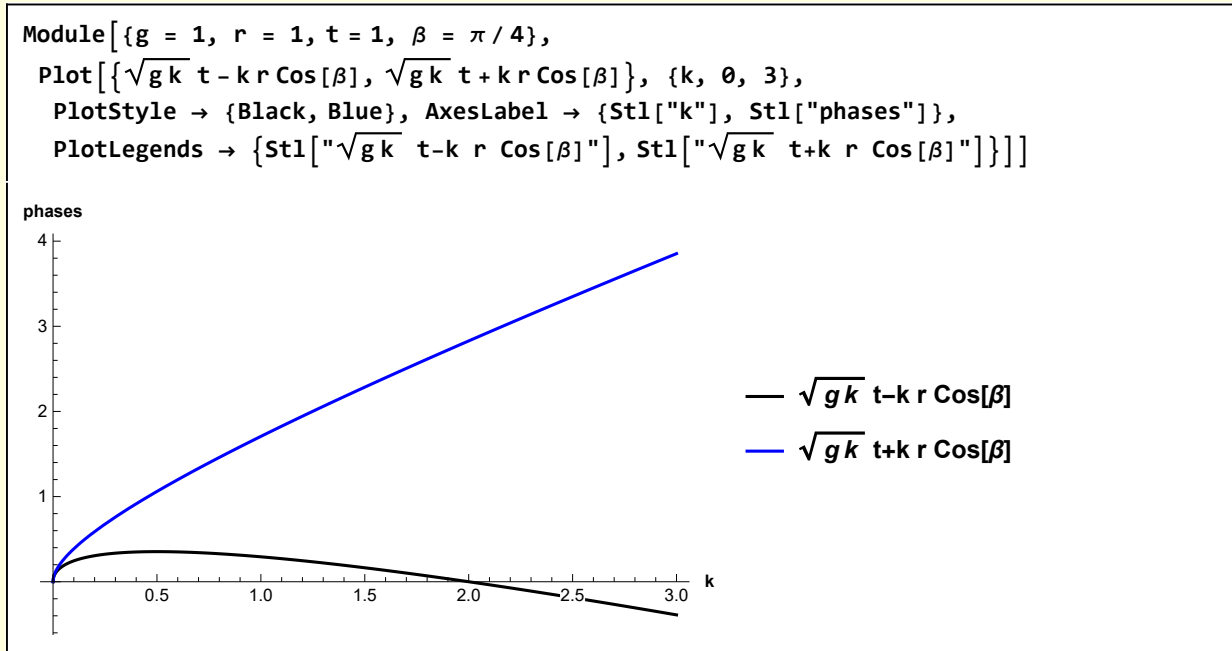
$$-\left(\text{Int}\beta\left[\text{Intk}\left[e^{i\left(\sqrt{gk}t - kr\cos[\beta]\right)}k^{3/2}\right]\right]\right) / \left(2\sqrt{g}\pi^2\rho\right) - \frac{\text{Int}\beta\left[\text{Intk}\left[e^{i\left(\sqrt{gk}t + kr\cos[\beta]\right)}k^{3/2}\right]\right]}{2\sqrt{g}\pi^2\rho}$$

The inner IntK integrals now have the appropriate form for the method of stationary phase. Recall

$$\text{Intk}\left(k^{3/2}e^{i\left(t\sqrt{gk} - kr\cos(\beta)\right)}\right) \equiv \int_0^\infty k^{3/2}e^{i\left(t\sqrt{gk} - kr\cos(\beta)\right)}dk$$

Note that there is no obvious large parameter  $p$ . I discuss this at the end of section 2.

For the application at hand,  $g > 0$ ,  $r > 0$ ,  $t > 0$ ,  $\cos(\beta) \geq 0$ . Only the first term has a point of stationary phase.



After these preliminary manipulations, the double integral to be performed is

$$\mathbf{wA1[9]} = \mathbf{wA1[8][[1]]} == \mathbf{wA1[8][[2, 1]]}$$

$$\delta z[r, t] == - \frac{\text{Int}\beta[\text{Int}k[e^{i(\sqrt{gk} t - kr \cos[\beta])} k^{3/2}]]}{2 \sqrt{g} \pi^2 \rho}$$

$$\mathbf{w1A1["\delta z[r,t] as double integral"]} = \mathbf{wA1[9]}$$

$$\delta z[r, t] == - \left( \text{Int}\beta[\text{Int}k[e^{i(\sqrt{gk} t - kr \cos[\beta])} k^{3/2}]] \right) / (2 \sqrt{g} \pi^2 \rho)$$

It should be recalled that it is the imaginary part of the rhs that is physically relevant.

## A1 Evaluation of the k-integral using method of stationary phase

The k-integral to be evaluated is

$$\mathbf{w1A1[1]} = \text{Int}k[e^{i(\sqrt{gk} t - kr \cos[\beta])} k^{3/2}] \equiv \text{Int}k[e^{i f[k]} g[k]]$$

$$\text{Int}k[e^{i(\sqrt{gk} t - kr \cos[\beta])} k^{3/2}] \equiv \text{Int}k[e^{i f[k]} g[k]]$$

I could just invoke a formula at this point (see Appendix C), but it is more instructive to work through the details. Consider the rhs

```
w1A1[2] = w1A1[1][[2]] /. f[k] → Normal@Series[f[k], {k, ks, 2}] /. f'[ks] → 0 /.
g[k] → g[ks]
```

```
g[ks] Intk[ $e^{i \left( f[ks] + \frac{1}{2} (k-ks)^2 f''[ks] \right)}$ ]
```

Make the sign of the quadratic term explicit

```
w1A1[3] = w1A1[2] /. f''[ks] → S[f''[ks]] A[f''[ks]]
```

```
g[ks] Intk[ $e^{i \left( f[ks] + \frac{1}{2} (k-ks)^2 A[f''[ks]] S[f''[ks]] \right)}$ ]
```

where I temporarily use A instead of Abs and S instead of Sign to simplify the expression resulting from application of Integrate

```
Clear[Fk, Gk];
Fk[k_] :=  $\sqrt{g k} t - k r \text{Cos}[\beta]$ ;
Gk[k_] :=  $k^{3/2}$ ;
```

```
w1A1[4] = Sol[D[Fk[k], k] == 0, k] /. k → ks
```

```
ks →  $\frac{g t^2 \text{Sec}[\beta]^2}{4 r^2}$ 
```

```
w1A1[5] = f[ks] == Simplify[Fk[ks] /. w1A1[4], {g > 0, t > 0, r > 0, 0 <  $\beta$  <  $\pi/2$ }]
```

```
f[ks] ==  $\frac{g t^2 \text{Sec}[\beta]}{4 r}$ 
```

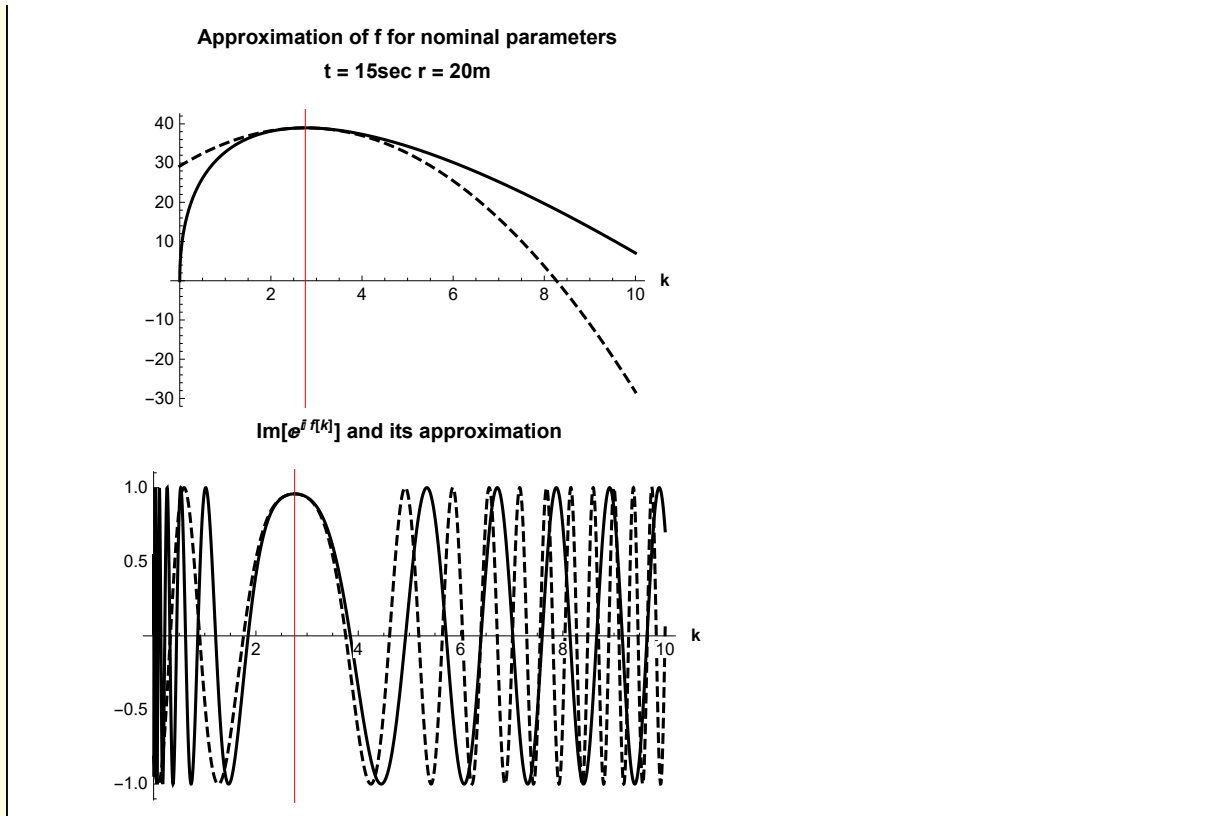
```
w1A1[6] = f''[ks] ==
Simplify[D[Fk[k], {k, 2}] /. k → ks /. w1A1[4], {g > 0, t > 0, r > 0, 0 <  $\beta$  <  $\pi/2$ }]
```

```
f''[ks] ==  $-\frac{2 r^3 \text{Cos}[\beta]^3}{g t^2}$ 
```

```

Module[{g = 9.8 (*m/sec^2*),  $\beta = \pi/4$ , t = 15, r = 20, kMax = 10,
  image = {400, 200}, ks, fks, d2fks, ksLine, lab, directives, G},
  Fk[k_] :=  $\sqrt{g k} t - k r \text{Cos}[\beta]$ ;
  Gk[k_] :=  $k^{3/2}$ ;
  ks =  $\frac{g t^2 \text{Sec}[\beta]^2}{4 r^2}$ ;
  fks =  $\frac{g t^2 \text{Sec}[\beta]}{4 r}$ ;
  d2fks =  $-\frac{2 r^3 \text{Cos}[\beta]^3}{g t^2}$ ;
  ksLine = {Red, Line[{{ks, -1000}, {ks, 1000}}]};
  lab = Stl@
    StringForm["Approximation of f for nominal parameters\n t = ``sec r = ``m", t, r];
  directives = {PlotStyle -> {Black, Directive[Black, Dashed]},
    AxesLabel -> {Stl["k"], Stl[""]}, ImageSize -> image};
  G[1] = Plot[{Fk[k], fks + d2fks  $\frac{(k - ks)^2}{2}$ }, {k, 0, kMax},
    PlotLabel -> lab, Epilog -> ksLine, Evaluate@directives];
  lab = Stl@StringForm["Im[ei f[k]] and its approximation"];
  G[2] = Plot[{Im[Exp[I Fk[k]]], Im[Exp[I (fks + d2fks  $\frac{(k - ks)^2}{2}$ )]]},
    {k, 0, kMax}, PlotLabel -> lab, Epilog -> ksLine, Evaluate@directives];
  Grid[{{G[1]}, {G[2]}}]
]

```



```
w1A1[7] = w1A1[3] /. Intk[a_] => Integrate[a, {k, -∞, ∞}]
```

```
ConditionalExpression[  
   $\frac{e^{i f[k_s]} \sqrt{2 \pi} g[k_s]}{\sqrt{-i A[f''[k_s]] S[f''[k_s]]}}$ , Im[A[f''[k_s]] S[f''[k_s]]] > 0]
```

```
w1A1[8] = Simplify[w1A1[7], Im[A[f''[k_s]] S[f''[k_s]]] > 0]
```

$$\frac{e^{i f[k_s]} \sqrt{2 \pi} g[k_s]}{\sqrt{-i A[f''[k_s]] S[f''[k_s]]}}$$

Note that

```
PolarForm[1 / √-I]
```

$$e^{\frac{i \pi}{4}}$$

```
w1A1[9] = w1A1[8] /.  $\frac{1}{\sqrt{-i A[f''[k_s]] S[f''[k_s]]}}$  →  $\frac{e^{\frac{i \pi}{4}} S[f''[k_s]]}{\sqrt{A[f''[k_s]]}}$ 
```

$$\frac{e^{i f[k_s] + \frac{1}{4} i \pi S[f''[k_s]]} \sqrt{2 \pi} g[k_s]}{\sqrt{A[f''[k_s]]}}$$

Then

$$\text{w1A1}[10] = \text{w1A1}[9] /. (\text{w1A1}[5] // \text{ER}) /. (\text{w1A1}[6] // \text{ER}) /. \text{g}[ks] \rightarrow \text{Gk}[ks] /. \text{w1A1}[4]$$

$$\left( e^{\frac{i}{4} \pi S \left[ -\frac{2r^3 \cos[\beta]^3}{g t^2} \right] + \frac{i g t^2 \text{Sec}[\beta]}{4r}} \sqrt{\frac{\pi}{2}} \left( \frac{g t^2 \text{Sec}[\beta]^2}{r^2} \right)^{3/2} \right) / \left( 4 \sqrt{A \left[ -\frac{2r^3 \cos[\beta]^3}{g t^2} \right]} \right)$$

$$\text{w1A1}[11] = \text{w1A1}[10] /. A[a_] \rightarrow \text{Abs}[a] /. S[a_] \rightarrow \text{Sign}[a]$$

$$\left( e^{\frac{i g t^2 \text{Sec}[\beta]}{4r} - \frac{i \pi \text{Sign}[r]^3 \text{Sign}[\cos[\beta]]^3}{4 \text{Sign}[g] \text{Sign}[t]^2}} \sqrt{\pi} \left( \frac{g t^2 \text{Sec}[\beta]^2}{r^2} \right)^{3/2} \right) / \left( 8 \sqrt{\text{Abs} \left[ \frac{r^3 \cos[\beta]^3}{g t^2} \right]} \right)$$

$$\text{w1A1}[12] = \text{Simplify}[\text{w1A1}[11], \{g > 0, r > 0, t > 0, \beta > 0, \cos[\beta] > 0, \text{Sec}[\beta] > 0\}]$$

$$-\frac{(-1)^{3/4} e^{\frac{i g t^2 \text{Sec}[\beta]}{4r}} g^2 \sqrt{\pi} t^4}{8 (r \cos[\beta])^{9/2}}$$

$$\text{w1A1}[13] = \text{w1A1}[12] /. (-1)^{3/4} \rightarrow \text{PolarForm} [(-1)^{3/4}] // \text{PowerExpand}$$

$$-\frac{e^{-\frac{i \pi}{4} + \frac{i g t^2 \text{Sec}[\beta]}{4r}} g^2 \sqrt{\pi} t^4}{8 r^{9/2} \cos[\beta]^{9/2}}$$

Finally

$$\text{w1A1}[14] = \text{w1A1}[1][[1]] == \text{w1A1}[13]$$

$$\text{Intk} \left[ e^{i \left( \sqrt{gk} t - kr \cos[\beta] \right)} k^{3/2} \right] == -\frac{e^{-\frac{i \pi}{4} + \frac{i g t^2 \text{Sec}[\beta]}{4r}} g^2 \sqrt{\pi} t^4}{8 r^{9/2} \cos[\beta]^{9/2}}$$

$$\text{w1A1}["k\text{-integral}"] = \text{w1A1}[14]$$

$$\text{Intk} \left[ e^{i \left( \sqrt{gk} t - kr \cos[\beta] \right)} k^{3/2} \right] == -\frac{e^{-\frac{i \pi}{4} + \frac{i g t^2 \text{Sec}[\beta]}{4r}} g^2 \sqrt{\pi} t^4}{8 r^{9/2} \cos[\beta]^{9/2}}$$

## A2 Evaluation of the $\beta$ -integral using method of stationary phase

Recall the double integral form of  $\delta z$  and use the result of the previous section



w1A2[1] = w1A1["δz[r,t] as double integral"] /. (w1A1["k-integral"] // ER)

$$\delta z[r, t] = \frac{g^{3/2} t^4 \text{Int}\beta \left[ \frac{e^{-\frac{i\pi}{4} - \frac{i g t^2 \text{Sec}[\beta]}{4r}}}{\text{Cos}[\beta]^{9/2}} \right]}{16 \pi^{3/2} r^{9/2} \rho}$$

Simplifying

$$w1A2[2] = \delta z[r, t] = \frac{g^{3/2} t^4 e^{-\frac{i\pi}{4}} \text{Int}\beta \left[ \frac{e^{\frac{i g t^2 \text{Sec}[\beta]}{4r}}}{\text{Cos}[\beta]^{9/2}} \right]}{16 \pi^{3/2} r^{9/2} \rho}$$

$$\delta z[r, t] = \frac{e^{-\frac{i\pi}{4}} g^{3/2} t^4 \text{Int}\beta \left[ \frac{e^{\frac{i g t^2 \text{Sec}[\beta]}{4r}}}{\text{Cos}[\beta]^{9/2}} \right]}{16 \pi^{3/2} r^{9/2} \rho}$$

w1A2["δz[r,t] as single integral"] = w1A2[2]

$$\delta z[r, t] = \frac{e^{-\frac{i\pi}{4}} g^{3/2} t^4 \text{Int}\beta \left[ \frac{e^{\frac{i g t^2 \text{Sec}[\beta]}{4r}}}{\text{Cos}[\beta]^{9/2}} \right]}{16 \pi^{3/2} r^{9/2} \rho}$$

Focus on the integral

$$w1A2[3] = \text{Int}\beta \left[ \frac{e^{\frac{i g t^2 \text{Sec}[\beta]}{4r}}}{\text{Cos}[\beta]^{9/2}} \right]$$

$$\text{Int}\beta \left[ \frac{e^{\frac{i g t^2 \text{Sec}[\beta]}{4r}}}{\text{Cos}[\beta]^{9/2}} \right]$$

This already has the correct form for application of the method of stationary phase

w1A2[4] =  
 $\text{Int}\beta \left[ e^{i f[\beta]} g[\beta] \right] /. f[\beta] \rightarrow \text{Normal@Series}[f[\beta], \{\beta, \beta s, 2\}] /. f'[\beta s] \rightarrow 0 /.$   
 $g[\beta] \rightarrow g[\beta s]$

$$g[\beta s] \text{Int}\beta \left[ e^{i \left( f[\beta s] + \frac{1}{2} (\beta - \beta s)^2 f''[\beta s] \right)} \right]$$

I just reuse the formula derived in the previous section

w1A2[5] = w1A1[9] /. ks → βs

$$\frac{e^{i f[\beta s] + \frac{1}{4} i \pi S[f''[\beta s]]} \sqrt{2 \pi} g[\beta s]}{\sqrt{A[f''[\beta s]]}}$$

Specifically

```
Clear[Fβ, Gβ];
Fβ[β_] :=  $\frac{g t^2 \text{Sec}[\beta]}{4 r}$ ;
Gβ[β_] :=  $1 / \text{Cos}[\beta]^{9/2}$ ;
```

```
w1A2[6] = Solve[D[Fβ[β], β] == 0, β][[1, 1]] /. β → β
β → ConditionalExpression[2 π C[1], C[1] ∈ Integers]
```

```
w1A2[7] = w1A2[6] /. C[1] → 0 /. β → βs
βs → 0
```

So

```
w1A2[8] =
{ f[βs] → Fβ[βs], g[βs] → Gβ[βs], {f''[βs] → D[Fβ[β], {β, 2}]} /. β → βs }
{ f[βs] →  $\frac{g t^2 \text{Sec}[\beta s]}{4 r}$ , g[βs] →  $\frac{1}{\text{Cos}[\beta s]^{9/2}}$ , f''[βs] →  $\frac{g t^2 (\text{Sec}[\beta s]^3 + \text{Sec}[\beta s] \text{Tan}[\beta s]^2)}{4 r}$  }
```

At the stationary point

```
w1A2[9] = w1A2[8] /. w1A2[7]
{ f[0] →  $\frac{g t^2}{4 r}$ , g[0] → 1, f''[0] →  $\frac{g t^2}{4 r}$  }
```

So

```
w1A2[10] =  $\frac{1}{2}$  w1A2[5] /. βs → 0 /. w1A2[9]
 $\frac{e^{i \frac{g t^2}{4 r} + \frac{1}{4} i \pi S\left[\frac{g t^2}{4 r}\right]} \sqrt{\frac{\pi}{2}}}{\sqrt{A\left[\frac{g t^2}{4 r}\right]}}$ 
```

where the factor 1/2 comes from the fact that the point of stationary phase occurs at  $\beta s = 0$  occurs at the endpoint of the integration range  $0 < \beta < \pi/2$

```
w1A2[11] = w1A2[10] /. A → Abs /. S → Sign
 $\frac{e^{i \frac{g t^2}{4 r} + \frac{i \pi \text{Sign}[g] \text{Sign}[t]^2}{4 \text{Sign}[r]}} \sqrt{2 \pi}}{\sqrt{\text{Abs}\left[\frac{g t^2}{r}\right]}}$ 
```

w1A2[12] = Simplify[w1A2[11], {g > 0, r > 0, t > 0}] // PowerExpand

$$\frac{(1 + i) e^{\frac{i g t^2}{4 r}} \sqrt{\pi} \sqrt{r}}{\sqrt{g} t}$$

w1A2[13] = w1A2[12] /. (1 + i) → PolarForm[1 + I]

$$\frac{e^{\frac{i \pi}{4} + \frac{i g t^2}{4 r}} \sqrt{2 \pi} \sqrt{r}}{\sqrt{g} t}$$

The calculation has resulted in

w1A2[14] = w1A2[3] == w1A2[13]

$$\text{Int}\beta\left[\frac{e^{\frac{i g t^2 \text{Sec}[\beta]}{4 r}}}{\text{Cos}[\beta]^{9/2}}\right] == \frac{e^{\frac{i \pi}{4} + \frac{i g t^2}{4 r}} \sqrt{2 \pi} \sqrt{r}}{\sqrt{g} t}$$

w1A2["β-integral"] = w1A2[14]

$$\text{Int}\beta\left[\frac{e^{\frac{i g t^2 \text{Sec}[\beta]}{4 r}}}{\text{Cos}[\beta]^{9/2}}\right] == \frac{e^{\frac{i \pi}{4} + \frac{i g t^2}{4 r}} \sqrt{2 \pi} \sqrt{r}}{\sqrt{g} t}$$

Using this in the expression for  $\delta z$  results in

w1A2["δz[r,t] as single integral"]

$$\delta z[r, t] == \frac{e^{-\frac{i \pi}{4}} g^{3/2} t^4 \text{Int}\beta\left[\frac{e^{\frac{i g t^2 \text{Sec}[\beta]}{4 r}}}{\text{Cos}[\beta]^{9/2}}\right]}{16 \pi^{3/2} r^{9/2} \rho}$$

w1A2[15] = w1A2["δz[r,t] as single integral"] /. (w1A2["β-integral"] // ER)

$$\delta z[r, t] == \frac{e^{\frac{i g t^2}{4 r}} g t^3}{8 \sqrt{2} \pi r^4 \rho}$$

Recall that it is the imaginary part of this that is of physical interest

w1A2[16] = w1A2[15] // ExpToTrig

$$\delta z[r, t] == \frac{g t^3 \text{Cos}\left[\frac{g t^2}{4 r}\right]}{8 \sqrt{2} \pi r^4 \rho} + \frac{i g t^3 \text{Sin}\left[\frac{g t^2}{4 r}\right]}{8 \sqrt{2} \pi r^4 \rho}$$

After a long and winding path the result in Equation (2) has been derived.

```
w1A2[17] = w1A2[16][[1]] == w1A2[16][[2, 2]]/I
```

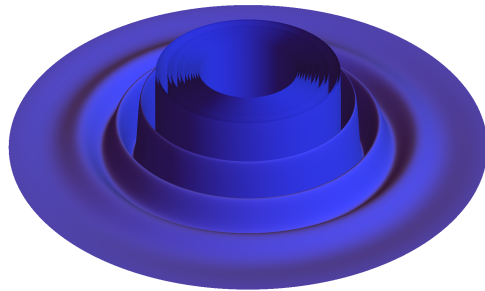
$$\delta z[r, t] == \frac{g t^3 \operatorname{Sin}\left[\frac{g t^2}{4 r}\right]}{8 \sqrt{2} \pi r^4 \rho}$$

```
w1A2["δz[r,t]"] = w1A2[17]
```

$$\delta z[r, t] == \frac{g t^3 \operatorname{Sin}\left[\frac{g t^2}{4 r}\right]}{8 \sqrt{2} \pi r^4 \rho}$$

I illustrate the spatial variation of this for some nominal parameters

```
Module[{g = 9.8 (*m/s^2*), rMin = 30, rMax = 150, tVal = 30, ϕ, Ψ, legends, lab, G},
  ϕ[r_, t_] :=  $\frac{g t^2}{4 r}$ ;
  Ψ[r_, t_] :=  $\frac{t^3}{r^4}$ ;
  RevolutionPlot3D[Ψ[r, tVal] Sin[ϕ[r, tVal]],
    {r, rMin, rMax}, PlotStyle → Lighter[Blue, 0.30],
    Mesh → False, Axes → False, Boxed → False, PlotPoints → 200]
```



## Appendix B: Explicit expression for the $s$ integral

Explicit forms for the terms in equation (6) are calculated. For the stationary phase approximation the following expansion was performed

$$\begin{aligned} \text{WB}[1] = \\ \hat{\Phi}[r, s] &= \left( \text{Normal}[\text{Series}[\hat{\Phi}[r, s], \{s, sP, 2\}] \right] /. \hat{\Phi}^{(0,1)}[r, sP] \rightarrow 0 /. r \rightarrow r[sP] \\ \hat{\Phi}[r, s] &= \hat{\Phi}[r[sP], sP] + \frac{1}{2} (s - sP)^2 \hat{\Phi}^{(0,2)}[r[sP], sP] \end{aligned}$$

where we have used  $\hat{\Phi}$  since  $\Phi$  already has a function assignment in the notebook.

Recall from Section 2 that the stationary phase condition is

$$\text{WB}[2] = \text{w2["circle of influence"]}$$

$$r[s] = \frac{1}{2} s V \text{Cos}[\alpha[s]]$$

Thus

$$\text{WB}[3] = \hat{\Phi}[r[sP], sP] = \bar{\Phi}[r[s], s] /. (\text{WB}[2] // \text{ER})$$

$$\hat{\Phi}[r[sP], sP] = \frac{g s \text{Sec}[\alpha[s]]}{2 V}$$

The second derivative requires some work

$$\text{WB}[4] = \hat{\Phi}^{(0,2)}[r[sP], sP] = D[\bar{\Phi}[r[s], s], \{s, 2\}]$$

$$\hat{\Phi}^{(0,2)}[r[sP], sP] = \frac{1}{4} g \left( \frac{2}{r[s]} - \frac{4 s r'[s]}{r[s]^2} + s^2 \left( \frac{2 r'[s]^2}{r[s]^3} - \frac{r''[s]}{r[s]^2} \right) \right)$$

To evaluate this expression, recall

$$\text{w2["st phase ode"]}$$

$$r'[s] = \frac{2 r[s]}{s}$$

$$\text{w2["st phase r ode"]}$$

$$r'[s] = V \text{Cos}[\alpha[s]]$$

$$\text{w2["st phase } \alpha \text{ ode"]}$$

$$\alpha'[s] \rightarrow -\frac{V \text{Sin}[\alpha[s]]}{r[s]}$$

**wB[5] = wB[4] /. (w2["st phase ode"] // ER) // Expand**

$$\hat{\Phi}^{(0,2)}[r[sP], sP] == \frac{g}{2 r[s]} - \frac{g s^2 r''[s]}{4 r[s]^2}$$

It remains to calculate the second derivative of r(s) along the path of stationary phase

**wB[6] = MapEqn[D[#, s] &, w2["st phase r ode"]]**

$$r''[s] == -V \text{Sin}[\alpha[s]] \alpha'[s]$$

**wB[7] = wB[6] /. w2["st phase  $\alpha$  ode"]**

$$r''[s] == \frac{V^2 \text{Sin}[\alpha[s]]^2}{r[s]}$$

or

**wB[8] = MapEqn[(# / r[s]) &, wB[7]]**

$$\frac{r''[s]}{r[s]} == \frac{V^2 \text{Sin}[\alpha[s]]^2}{r[s]^2}$$

Make the rhs explicit

**wB[9] = wB[8][[1]] == (wB[8][[2]] /. (wB[2] // ER))**

$$\frac{r''[s]}{r[s]} == \frac{4 \text{Tan}[\alpha[s]]^2}{s^2}$$

Finally

**wB[10] = wB[5] /. Sol[wB[9], r''[s]]**

$$\hat{\Phi}^{(0,2)}[r[sP], sP] == \frac{g}{2 r[s]} - \frac{g \text{Tan}[\alpha[s]]^2}{r[s]}$$

The angular dependence is

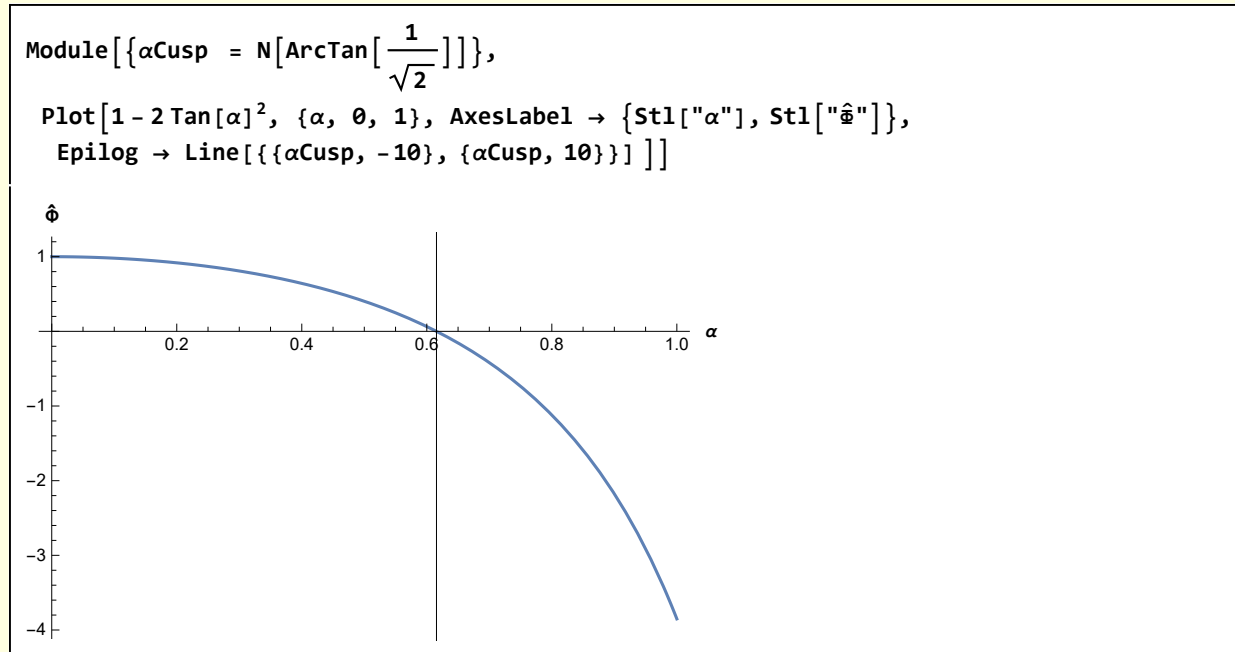
$$\text{wB[11]} = 1 - 2 \text{Tan}[\alpha]^2$$

$$1 - 2 \text{Tan}[\alpha]^2$$

**wB[12] = Solve[wB[11] == 0,  $\alpha$ ][[2, 1]] /. C[1] → 0 /.  $\alpha$  →  $\alpha_{\text{cusp}}$**

$$\alpha_{\text{cusp}} \rightarrow \text{ArcTan}\left[\frac{1}{\sqrt{2}}\right]$$

This angle was encountered in Section 2 and was associated with the abrupt change of phase front geometry.



Gather the results

```
wB["ϕ̂ results"] = {wB[3], wB[10],
sign[ϕ̂(0,2)[r[sP], sP]] == (α[s] < αCusp), sign[ϕ̂(0,2)[r[sP], sP]] == -(α[s] > αCusp)}
{ϕ̂[r[sP], sP] ==  $\frac{g s \text{Sec}[\alpha[s]]}{2 V}$ , ϕ̂(0,2)[r[sP], sP] ==  $\frac{g}{2 r[s]} - \frac{g \text{Tan}[\alpha[s]]^2}{r[s]}$ ,
sign[ϕ̂(0,2)[r[sP], sP]] == (α[s] < αCusp), sign[ϕ̂(0,2)[r[sP], sP]] == -(α[s] > αCusp)}
```

## Appendix C Derivation of Stationary Phase Formula

I present a quick derivation of the stationary phase approximation to illustrate one way the requisite steps could be performed using Mathematica operations. A classical example is also considered

The canonical result for the method of stationary phase is

$$J = \int_{-\infty}^{\infty} dx \psi(x) e^{i p \phi(x)} \simeq \sqrt{\frac{2\pi}{k}} \frac{e^{i k \phi(x_0)} e^{\frac{i\pi}{4} \text{sign}(\phi''(x_0))}}{\sqrt{|\phi''(x_0)|}} \psi(x_0)$$

where  $x_0$  satisfies  $\phi'(x_0) = 0$

Define the integral



$$\text{wC}[1] = J == \text{Int}[\psi[x] \text{Ex}[\text{I p } \phi[x]]]$$

$$J == \text{Int}[\text{Ex}[\text{i p } \phi[x]] \psi[x]]$$

where Int and Ex are non-evaluating versions of Integrate and Exp introduced for convenience

Expand  $\phi$  about the stationary point  $x_0$

$$\text{wC}[2] = \text{wC}[1] /. \phi[x] \rightarrow \text{Normal}[\text{Series}[\phi[x], \{x, x_0, 2\}]] /. \psi[x] \rightarrow \psi[x_0]$$

$$J == \text{Int}[\text{Ex}[\text{i p} \left( \phi[x_0] + (x - x_0) \phi'[x_0] + \frac{1}{2} (x - x_0)^2 \phi''[x_0] \right)]] \psi[x_0]$$

Impose the condition for stationarity

$$\text{wC}[3] = \text{wC}[2] /. \phi'[x_0] \rightarrow 0$$

$$J == \text{Int}[\text{Ex}[\text{i p} \left( \phi[x_0] + \frac{1}{2} (x - x_0)^2 \phi''[x_0] \right)]] \psi[x_0]$$

and perform some simplifying operations

$$\text{wC}[4] =$$

$$\text{wC}[3] /. \text{Ex}[\text{i k} \left( \phi[x_0] + \frac{1}{2} (x - x_0)^2 \phi''[x_0] \right)] \rightarrow \text{Ex}[\text{I k } \phi[x_0]] \text{Ex}[\text{I k } \frac{1}{2} (x - x_0)^2 \phi''[x_0]]$$

$$J == \text{Int}[\text{Ex}[\text{i p} \left( \phi[x_0] + \frac{1}{2} (x - x_0)^2 \phi''[x_0] \right)]] \psi[x_0]$$

$$\text{wC}[5] = \text{wC}[4] //. \text{Int}[a\_b_] /; \text{FreeQ}[a, x] \rightarrow a \text{Int}[b]$$

$$J == \text{Int}[\text{Ex}[\text{i p} \left( \phi[x_0] + \frac{1}{2} (x - x_0)^2 \phi''[x_0] \right)]] \psi[x_0]$$

Replace Int and Ex

$$\text{temp} = \text{wC}[5][[2]] /. \text{Ex} \rightarrow \text{Exp} /. \text{Int}[a_] \Rightarrow \text{Integrate}[a, \{x, -\infty, \infty\}]$$

$$\text{ConditionalExpression}\left[\frac{e^{i p \phi[x_0]} \sqrt{2 \pi} \psi[x_0]}{\sqrt{-i p \phi''[x_0]}}, \text{Im}[p \phi''[x_0]] > 0\right]$$

$$\text{wC}[6] = \text{wC}[5][[1]] == \text{Simplify}[\text{temp}, \text{Im}[p \phi''[x_0]] > 0]$$

$$J == \frac{e^{i p \phi[x_0]} \sqrt{2 \pi} \psi[x_0]}{\sqrt{-i p \phi''[x_0]}}$$

Note

```
PolarForm[1 /  $\sqrt{-I}$ ]
```

$$e^{\frac{i\pi}{4}}$$

and make a simplifying substitution

```
wC[7] = wC[6] /. 1 /  $\sqrt{-i p \phi''[x0]}$   $\rightarrow$  Exp[I Sign[ $\phi''[x0]$ ]  $\pi / 4$ ] / ( $\sqrt{p} \sqrt{\text{Abs}[\phi''[x0]]}$ )
```

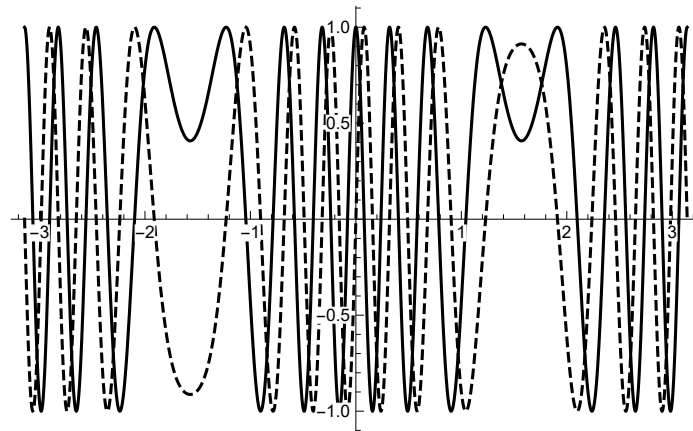
$$J = \frac{e^{\frac{1}{4} i \pi \text{Sign}[\phi''[x0]] + i p \phi[x0]} \sqrt{2 \pi} \psi[x0]}{\sqrt{p} \sqrt{\text{Abs}[\phi''[x0]]}}$$

which is the desired result.

A classic example for the method of stationary phase is the approximation of the Bessel function

$$J_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx e^{i p \sin[x]}$$

```
Module[{p = 20},
  Plot[{Re[Exp[I p Sin[x]]], Im[Exp[I p Sin[x]]]},
    {x, - $\pi$ ,  $\pi$ }, PlotStyle  $\rightarrow$  {Black, Directive[Black, Dashed]}]]
```



For the integral  $J_0$

```
Clear[ $\phi$ C,  $\Psi$ C];
 $\phi$ C[x_] := Sin[x];
 $\Psi$ C[x_] :=  $\frac{1}{2\pi}$ ;
```

There are two stationary points

$$\text{wC}[8] = \text{Solve}[D[\text{ϕC}[x], x] == 0, x] /. C[1] \to \theta /. x \to x\theta$$

$$\left\{ \left\{ x\theta \rightarrow -\frac{\pi}{2} \right\}, \left\{ x\theta \rightarrow \frac{\pi}{2} \right\} \right\}$$

$$\text{wC}[9] = \{\text{ϕC}[x], D[\text{ϕC}[x], \{x, 2\}]\} /. x \to x\theta$$

$$\{\text{Sin}[x\theta], -\text{Sin}[x\theta]\}$$

The contribution from the two stationary points are

$$\text{wC}[10] =$$

$$\text{Simplify}\left[\left(e^{\frac{1}{2}i\pi \text{Sign}[\phi''[x\theta]] + i p \phi[x\theta]} \sqrt{2\pi} \psi[x\theta]\right) / \left(\sqrt{p} \sqrt{\text{Abs}[\phi''[x\theta]]}\right) /. \{\psi[x\theta] \rightarrow \frac{1}{2\pi}, \phi[x\theta] \rightarrow \text{ϕC}[x\theta], \phi''[x\theta] \rightarrow D[\text{ϕC}[x\theta], \{x\theta, 2\}]\} /. x\theta \rightarrow -\pi/2, k > \theta\right] +$$

$$\text{Simplify}\left[\left(e^{\frac{1}{2}i\pi \text{Sign}[\phi''[x\theta]] + i p \phi[x\theta]} \sqrt{2\pi} \psi[x\theta]\right) / \left(\sqrt{p} \sqrt{\text{Abs}[\phi''[x\theta]]}\right) /. \{\psi[x\theta] \rightarrow \frac{1}{2\pi}, \phi[x\theta] \rightarrow \text{ϕC}[x\theta], \phi''[x\theta] \rightarrow D[\text{ϕC}[x\theta], \{x\theta, 2\}]\} /. x\theta \rightarrow \pi/2, k > \theta\right]$$

$$\frac{\left(\frac{1}{2} + \frac{i}{2}\right) e^{-ip}}{\sqrt{p} \sqrt{\pi}} + \frac{\left(\frac{1}{2} - \frac{i}{2}\right) e^{ip}}{\sqrt{p} \sqrt{\pi}}$$

or

$$\text{wC}[11] = \text{wC}[10] /. \left(\frac{1}{2} + \frac{i}{2}\right) \rightarrow \text{PolarForm}\left[\left(\frac{1}{2} + \frac{i}{2}\right)\right] /. \left(\frac{1}{2} - \frac{i}{2}\right) \rightarrow \text{PolarForm}\left[\left(\frac{1}{2} - \frac{i}{2}\right)\right]$$

$$\frac{e^{ip - \frac{i\pi}{4}}}{\sqrt{p} \sqrt{2\pi}} + \frac{e^{-ip + \frac{i\pi}{4}}}{\sqrt{p} \sqrt{2\pi}}$$

$$\text{wC}[12] = \text{wC}[11] // \text{ExpToTrig}$$

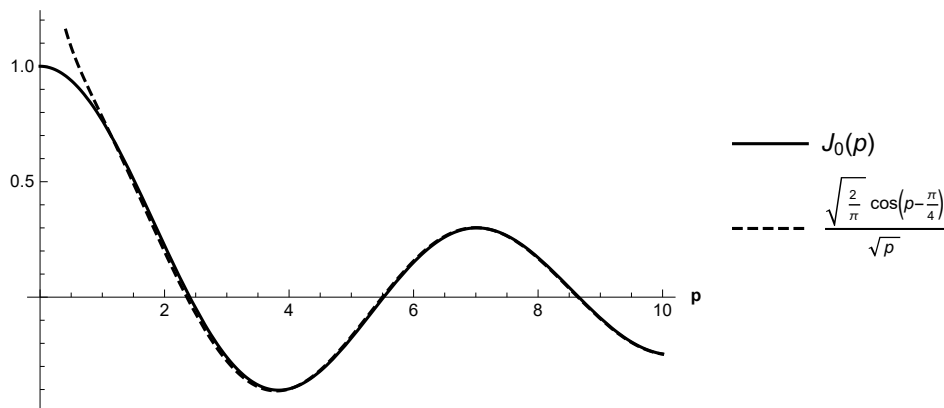
$$\frac{\sqrt{\frac{2}{\pi}} \text{Cos}\left[p - \frac{\pi}{4}\right]}{\sqrt{p}}$$

This approximation is accurate even for modest p

```

Plot[{{BesselJ[0, p],  $\frac{\sqrt{\frac{2}{\pi}} \cos[p - \frac{\pi}{4}]}{\sqrt{p}}$ }},
 {p, 0, 10}, PlotStyle -> {Black, Directive[Black, Dashed]},
 PlotLegends -> {{TraditionalForm[BesselJ[0, p]], TraditionalForm[ $\frac{\sqrt{\frac{2}{\pi}} \cos[p - \frac{\pi}{4}]}{\sqrt{p}}$ ]},
 AxesLabel -> {Stl["p"], Stl[""]}

```



## Figures

I define some functions that will be useful for the operations below

```

Clear[QPoint, QArc2D, QxAxis, QyAxis];
(* Label a point with offset text *)
QPoint[P_, text_, {lr_, ud_}] :=
  {Point[P], Text[Stl[text], P + {lr, ud}]};
QArc2D[r_, θS_, θF_, text_] :=
  {Line@Table[CoordinateTransform["Polar" -> "Cartesian", {r, ξ}],
    {ξ, θS, θF, Sign[θF - θS]  $\frac{\pi}{64}$ }],
  {Black, Text[Stl[text], CoordinateTransform[
    "Polar" -> "Cartesian", {1.25 r,  $\frac{1}{2}(\theta S + \theta F)$ ]}]}];
QxAxis[xStart_, xFin_, yLevel_, text_, {lr_, ud_}] :=
  {Line[{{xStart, yLevel}, {xFin, yLevel}}],
  Text[Stl[text], {xFin, yLevel} + {lr, ud}]};
QyAxis[yStart_, yFin_, xLevel_, text_, {lr_, ud_}] :=
  {Line[{{xLevel, yStart}, {xLevel, yFin}}],
  Text[Stl[text], {xLevel, yFin} + {lr, ud}]}

```

Figure 1

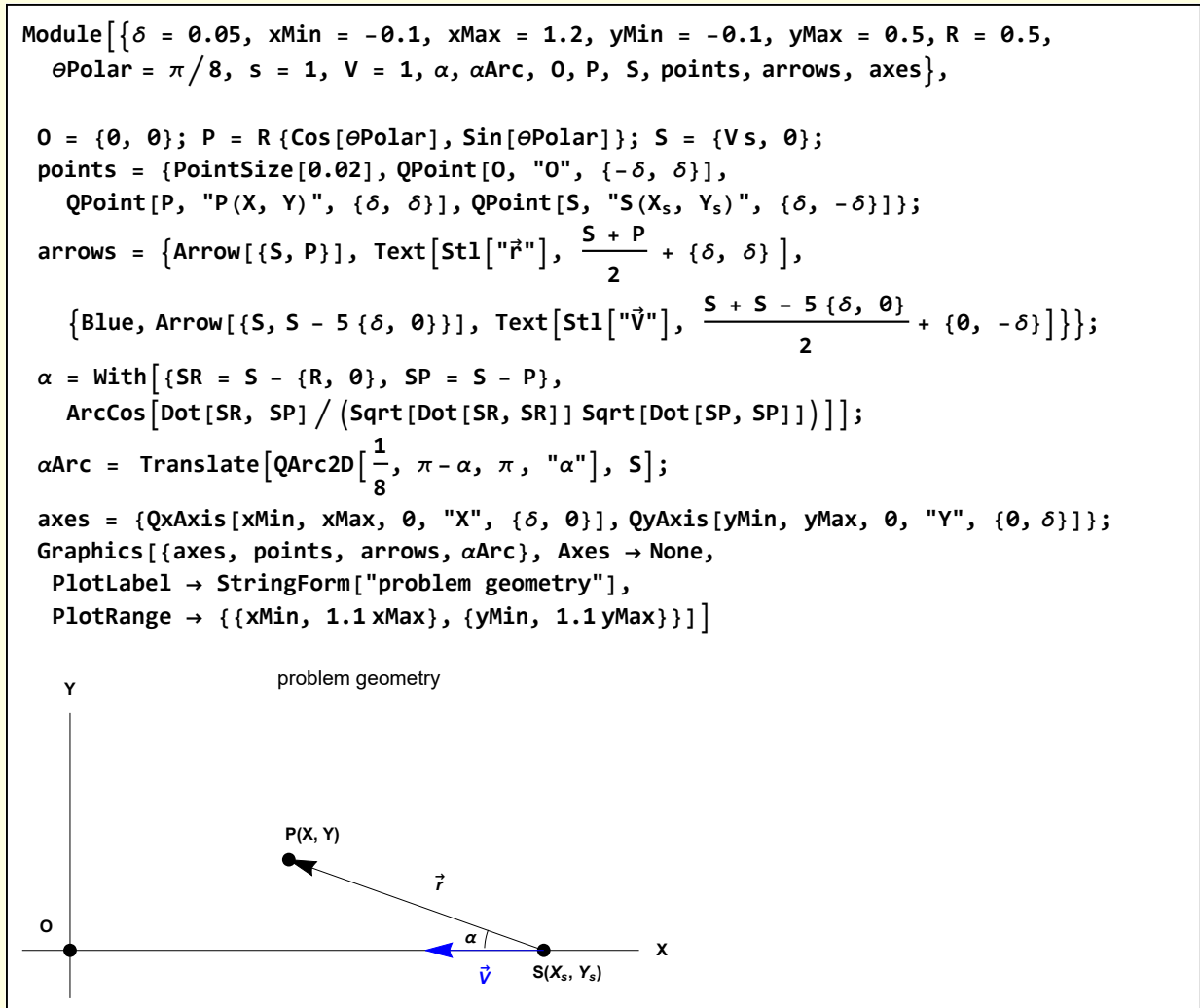
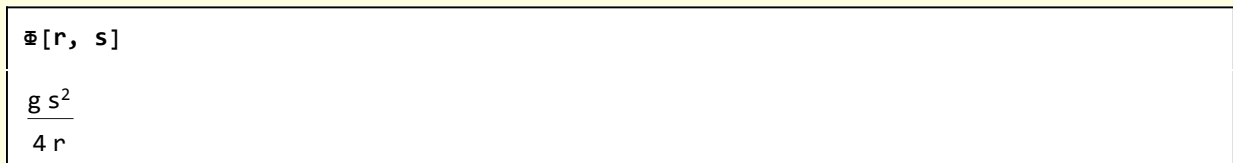


Figure 2



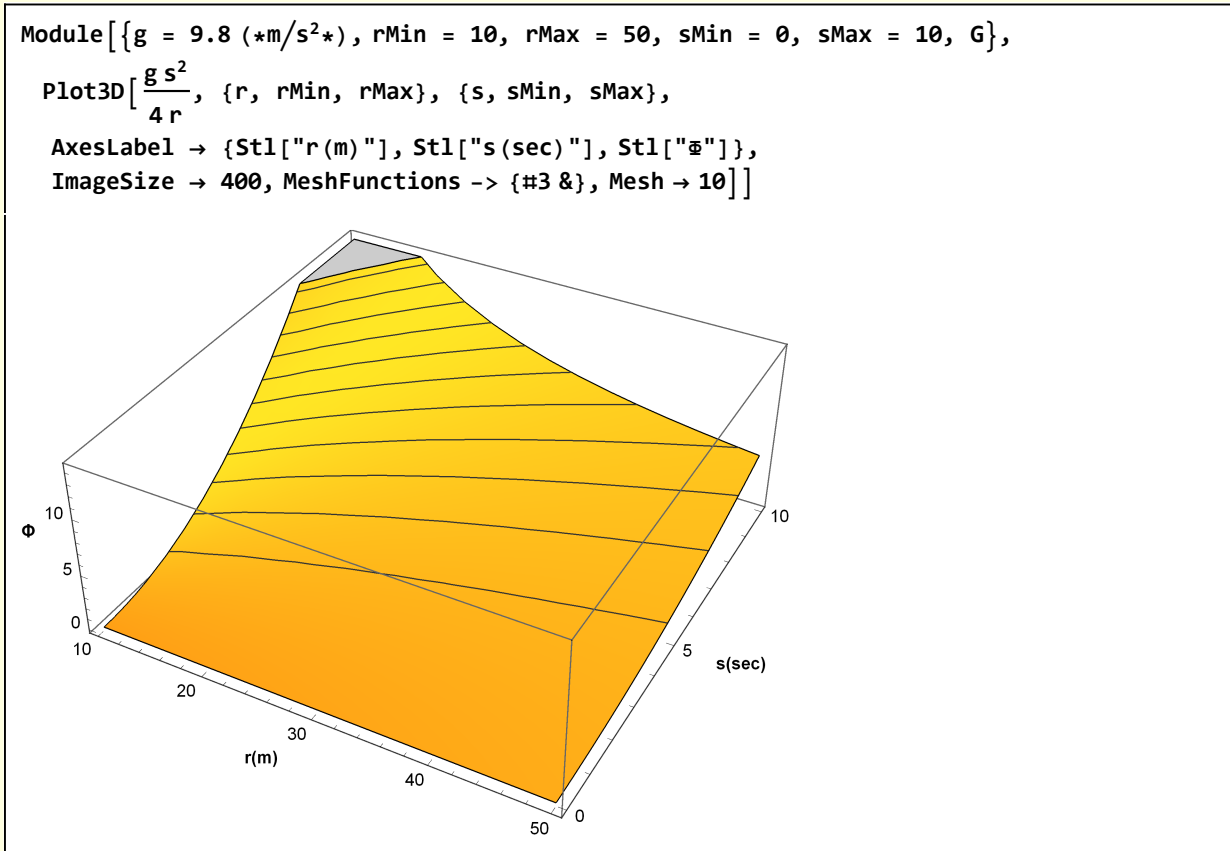
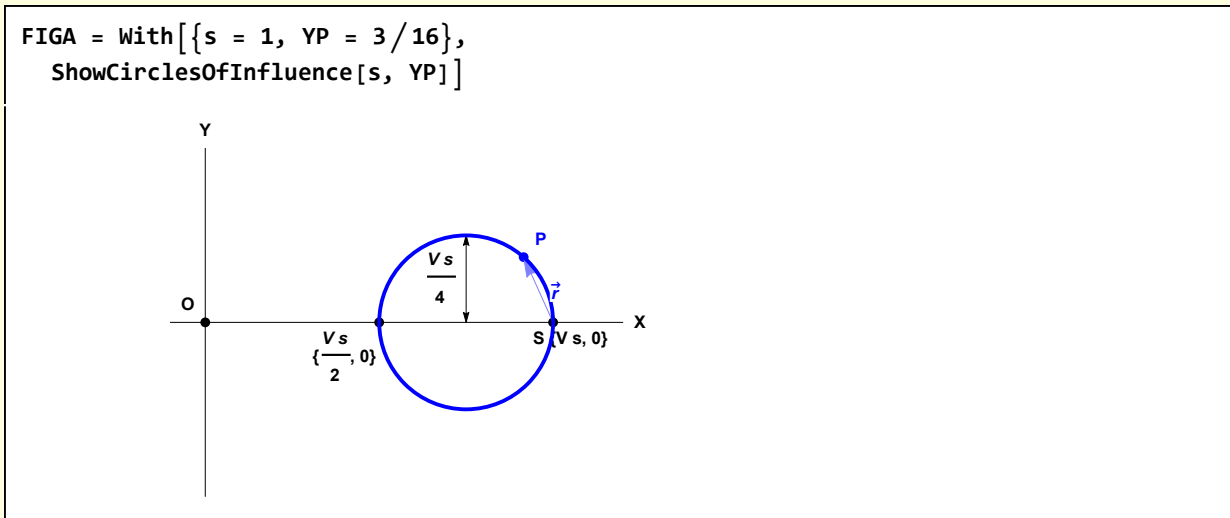
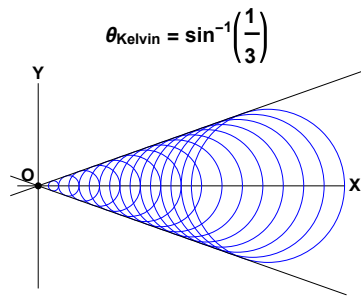


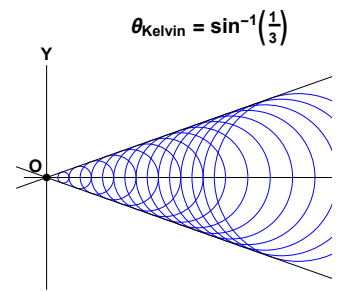
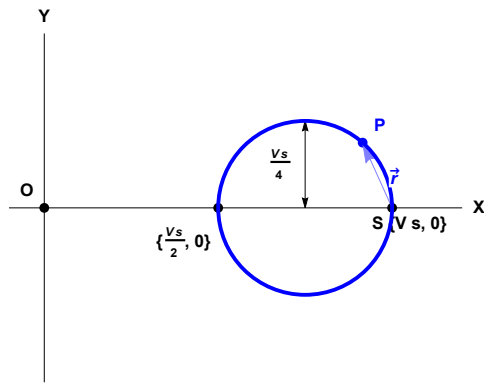
Figure 3



FIGB = With[{V = 1},  
LocusOfCircles[V]]



Grid[{{FIGA, FIGB}}]



```

Clear[ShowCirclesOfInfluence];
ShowCirclesOfInfluence[s_, YP_] :=
Module[{ $\delta = 0.05$ , V = 1, 0, S, P, xCenter,
  X0, RCOI, XP1, XP2, axes, points, lines, circle, labels},
  0 = {0, 0};
  S = {Vs, 0};
  (* The diameter of the circle of influence is V s/2 so the radius is *)
  RCOI = Vs / 4;
  (* The center of the circle of influence is 3 V s/4*)
  X0 = 3 Vs / 4;
  {XP1, XP2} = {X0 -  $\sqrt{-YP^2 + RCOI^2}$ , X0 +  $\sqrt{-YP^2 + RCOI^2}$ };
  P = {XP2, YP};

  points =
  {PointSize[0.02], QPoint[0, "0", {- $\delta$ ,  $\delta$ ], QPoint[S, "S {V s, 0}", { $\delta$ , - $\delta$ ],
  QPoint[S/2, " $\frac{Vs}{2}$ , 0", {-2 $\delta$ , -2 $\delta$ ], {Blue, QPoint[P, "P", { $\delta$ ,  $\delta$ ]}];

  lines = {{Lighter[Blue, 0.5], Arrow[{S, P}]},
  {Black, Arrowheads[{-0.02, 0.02}], Arrow[{{ $\frac{3Vs}{4}$ , 0}, { $\frac{3Vs}{4}$ ,  $\frac{Vs}{4}$ }]}];

  axes = {QxAxis[-0.1, 1.2, 0, "X", { $\delta$ , 0}], QyAxis[-0.5, 0.5, 0, "Y", {0,  $\delta$ ]}];
  circle = {Blue, Thick, Circle[3 S/4, Vs / 4];}
  labels = {{Blue, Text[Stl["r"],  $\frac{S + P}{2} + \{\delta, 0\}$ ]},
  {Black, Text[Stl[" $\frac{Vs}{4}$ "], { $\frac{3Vs}{4}$ ,  $\frac{Vs}{8}$ } + {-3/2 $\delta$ , 0}]}];

  Graphics[{axes, points, lines, circle, labels},
  Axes  $\rightarrow$  None, ImageSize  $\rightarrow$  {400, 200}]

```

```

Clear[LocusOfCircles];
LocusOfCircles[V_] :=
Module[{ $\delta = 0.05$ , circles, 0, S, axes, points,  $\theta_{\text{Kelvin}}$ , lab},
  0 = {0, 0};
  S = {V, 0};
   $\theta_{\text{Kelvin}} = \text{ArcSin}[1/3]$ ;
  circles = {Blue, Table[Circle[{{ $\frac{3Vs}{4}$ , 0},  $\frac{Vs}{4}$ ], {s, 0.1, 1.5, 0.1}]}];
  axes = {QxAxis[-0.1, 1.5, 0, "X", { $\delta$ , 0}], QyAxis[-0.5, 0.5, 0, "Y", {0,  $\delta$ ]}];
  points = {PointSize[0.02], QPoint[0, "0", {- $\delta$ ,  $\delta$ ]}];
  lab = Stl@StringForm[" $\theta_{\text{Kelvin}} = \text{``}$ ",  $\theta_{\text{Kelvin}}$ ];
  Graphics[{axes, points, circles,
  Infiniteline[{0, {1, Tan[ $\theta_{\text{Kelvin}}$ ]}}, Infiniteline[{0, {1, -Tan[ $\theta_{\text{Kelvin}}$ ]}]}],
  PlotLabel  $\rightarrow$  lab, Axes  $\rightarrow$  None, ImageSize  $\rightarrow$  {300, 150}]

```

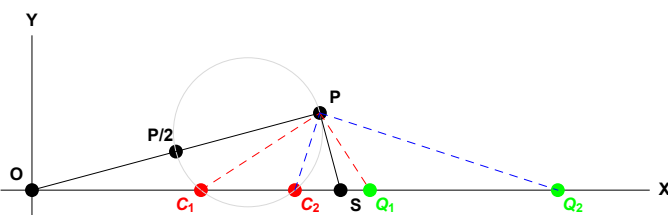


Figure 4

```

Module[{δ = 0.05, V = 1, s = 1, xB, yB, xC, yC,
  O, S, P, RB, circle, C1, C2, Q1, Q2, axes, points, lines},
  O = {0, 0};
  S = {Vs, 0};
  yB = 1/4;
  xB = With[{xA = 0 [[1]], xC = S [[1]]}, SolnProblem1[xA, xC, yB] [[1]];
  P = {xB, yB};
  RB = Norm[P] / 4;
  {xC, yC} = 3 P / 4;
  circle = Circle[3 P / 4, RB];
  {C1, C2} = {{xC - √(RB² - yC²), 0}, {xC + √(RB² - yC²), 0}};
  Q1 = {SolnProblem2[C1[[1]], P [[1]], P [[2]], 0};
  Q2 = {SolnProblem2[C2[[1]], P [[1]], P [[2]], 0};
  points = {PointSize[0.02], QPoint[O, "O", {-δ, δ}],
    QPoint[S, "S", {δ, -δ}], QPoint[P, "P", {δ, δ}], QPoint[P/2, "P/2", {-δ, δ}],
    {Red, QPoint[C1, "C1", {-δ, -δ}], QPoint[C2, "C2", {δ, -δ}]},
    {Green, QPoint[Q1, "Q1", {δ, -δ}], QPoint[Q2, "Q2", {δ, -δ]}}];
  lines = {Line[{O, P}], Line[{P, S}],
    {Directive[Red, Dashed], Line[{C1, P}], Line[{P, Q1}]},
    {Directive[Blue, Dashed], Line[{C2, P}], Line[{P, Q2]}}];
  (*Print[points];*)
  axes = {QxAxis[-0.1, 2.0, 0, "X", {δ, 0}], QyAxis[-0.1, 0.5, 0, "Y", {0, δ}];
  Graphics[{axes, points, {LightGray, circle}, lines}, Axes → None ]

```



### Some analytical geometry problems

Problem 1: Given A and C on the x-Axis, Find B such that AB and BC are orthogonal.  $y_B$  is given

```

p1[1] = {y == mAB x + bAB /. {x → xA, y → yA}, y == mAB x + bAB /. {x → xB, y → yB},
  y == mBC x + bBC /. {x → xB, y → yB}, y == mBC x + bBC /. {x → xC, y → yC}}

```

```

{yA == bAB + mAB xA, yB == bAB + mAB xB, yB == bBC + mBC xB, yC == bBC + mBC xC}

```

```

p1[2] = p1[1] /. {yA → 0, yC → 0, mBC → -1/mAB}

```

```

{0 == bAB + mAB xA, yB == bAB + mAB xB, yB == bBC - xB/mAB, 0 == bBC - xC/mAB}

```

`p1[3] = Solve[p1[2], {mAB, bAB, bBC, xB}]`

$$\left\{ \left\{ \begin{aligned} mAB &\rightarrow \frac{1}{2 y_B} \left( -x_A + x_C - \sqrt{x_A^2 - 2 x_A x_C + x_C^2 - 4 y_B^2} \right), \\ bAB &\rightarrow \frac{1}{2} \left( \frac{x_A^2}{y_B} - \frac{x_A x_C}{y_B} + \frac{x_A \sqrt{x_A^2 - 2 x_A x_C + x_C^2 - 4 y_B^2}}{y_B} \right), \\ bBC &\rightarrow \frac{1}{2 y_B} \left( -x_A x_C + x_C^2 + x_C \sqrt{x_A^2 - 2 x_A x_C + x_C^2 - 4 y_B^2} \right), \\ xB &\rightarrow \frac{1}{2} \left( x_A + x_C + \sqrt{x_A^2 - 2 x_A x_C + x_C^2 - 4 y_B^2} \right) \right\}, \\ \left\{ \begin{aligned} mAB &\rightarrow \frac{1}{2 y_B} \left( -x_A + x_C + \sqrt{x_A^2 - 2 x_A x_C + x_C^2 - 4 y_B^2} \right), \\ bAB &\rightarrow \frac{1}{2} \left( \frac{x_A^2}{y_B} - \frac{x_A x_C}{y_B} - \frac{x_A \sqrt{x_A^2 - 2 x_A x_C + x_C^2 - 4 y_B^2}}{y_B} \right), \\ bBC &\rightarrow \frac{1}{2 y_B} \left( -x_A x_C + x_C^2 - x_C \sqrt{x_A^2 - 2 x_A x_C + x_C^2 - 4 y_B^2} \right), \\ xB &\rightarrow \frac{1}{2} \left( x_A + x_C - \sqrt{x_A^2 - 2 x_A x_C + x_C^2 - 4 y_B^2} \right) \right\} \right\} \end{aligned}$$

`Clear[SolnProblem1];`

`SolnProblem1[xA_, xC_, yB_] := (*for xB *)`

$$\left\{ \frac{1}{2} \left( x_A + x_C + \sqrt{x_A^2 - 2 x_A x_C + x_C^2 - 4 y_B^2} \right), \frac{1}{2} \left( x_A + x_C - \sqrt{x_A^2 - 2 x_A x_C + x_C^2 - 4 y_B^2} \right) \right\}$$

Problem 2: Given A on the x-Axis, and B such that AB and BC are orthogonal. Find C on the x-Axis

`p2[1] = Solve[p1[2], {mAB, bAB, bBC, xC}][[1]]`

$$\left\{ mAB \rightarrow -\frac{y_B}{x_A - x_B}, bAB \rightarrow \frac{x_A y_B}{x_A - x_B}, bBC \rightarrow \frac{-x_A x_B + x_B^2 + y_B^2}{y_B}, xC \rightarrow \frac{-x_A x_B + x_B^2 + y_B^2}{-x_A + x_B} \right\}$$

`Clear[SolnProblem2];`

`SolnProblem2[xA_, xB_, yB_] := (*for xC *)`

$$\frac{-x_A x_B + x_B^2 + y_B^2}{-x_A + x_B}$$

Figure 5

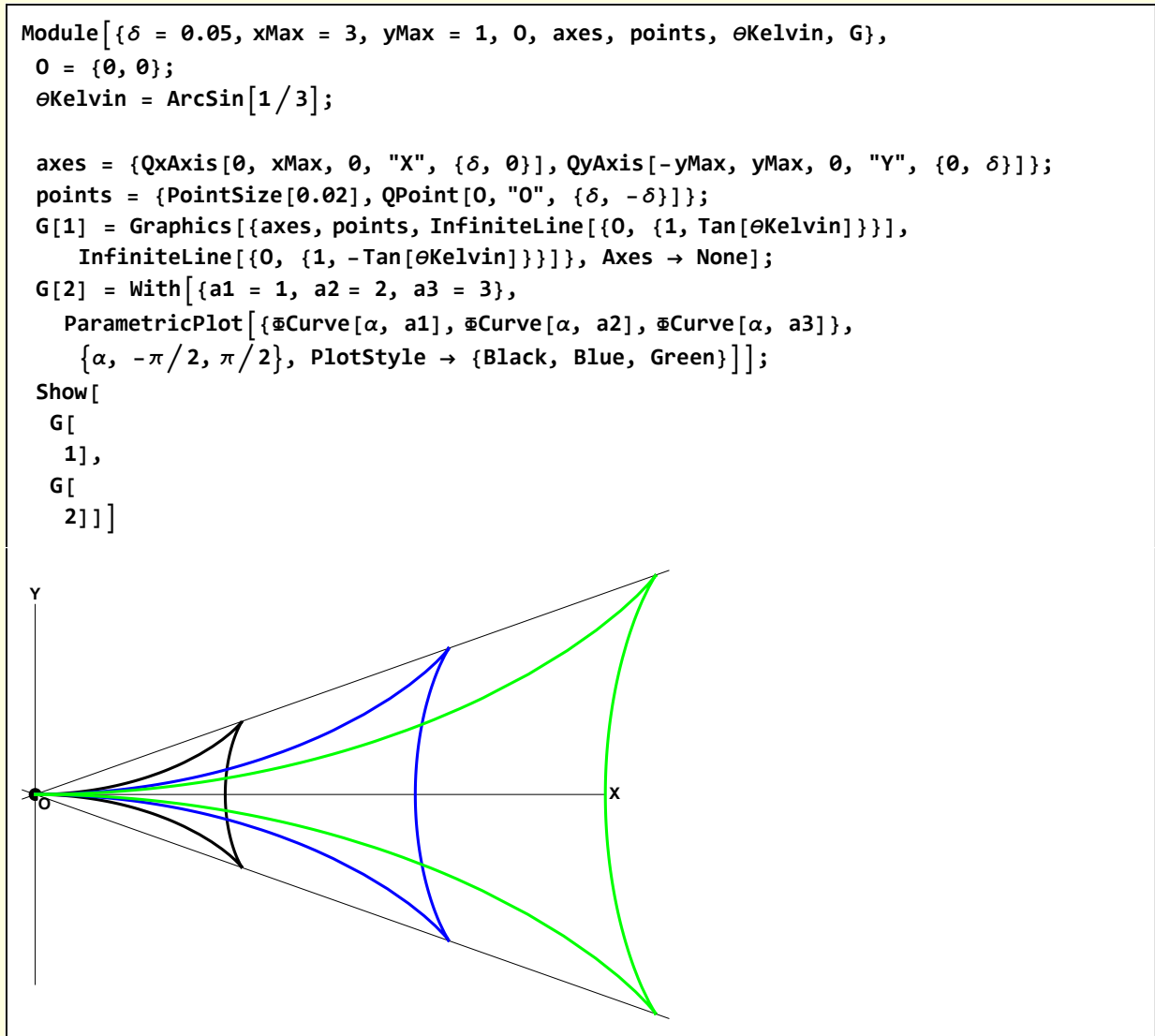


Figure 6

```

Module[{ $\delta = 0.075$ , a2 = 2, a3 = 3, xrange = {-0.2, 4.25},
  yRange = {-0.2, 1.2}, 0, P, Q1, Q2, axes, points, lines,  $\theta$ Kelvin,
  soln, XP, YP, mCurve1, mCurve2, directives, G, Stl2, QPoint2},
  Stl2[x_] := Style[x, 10, Bold, FontFamily  $\rightarrow$  "Helvetica"];
  QPoint2[P_, text_, {lr_, ud_}] := {Point[P], Text[Stl2[text], P + {lr, ud}]}];

  0 = {0, 0};
   $\theta$ Kelvin = ArcSin[1/3];
  (* Find the point P where the phase curves intersect *)
  soln = FindRoot[{ $\theta$ Curve[ $\alpha$ 1, a2][[1]] ==  $\theta$ Curve[ $\alpha$ 2, a3][[1]],
     $\theta$ Curve[ $\alpha$ 1, a2][[2]] ==  $\theta$ Curve[ $\alpha$ 2, a3][[2]]}, {{ $\alpha$ 1, 0.4}, { $\alpha$ 2, 1.3}}];
  P = {XP, YP} =  $\theta$ Curve[ $\alpha$ 1, a2] /. soln;

  (* find the slope of the phase curve at P *)
  (* Then the intersection of the normal line through P with the x-Axis *)
  mCurve1 = D[ $\theta$ Curve[ $\alpha$ 1, a2][[2]],  $\alpha$ 1] / D[ $\theta$ Curve[ $\alpha$ 1, a2][[1]],  $\alpha$ 1] /. soln;
  Q1 = {XP - YP / (-1/mCurve1), 0};
  mCurve2 = D[ $\theta$ Curve[ $\alpha$ 2, a3][[2]],  $\alpha$ 2] / D[ $\theta$ Curve[ $\alpha$ 2, a3][[1]],  $\alpha$ 2] /. soln;
  Q2 = {XP - YP / (-1/mCurve2), 0};

  axes = {Gray, QxAxis[0, xrange[[2]], 0, "X", { $\delta$ , 0}],
    QyAxis[-0.2, yRange[[2]], 0, "Y", {0,  $\delta$ ]}];
  points = {PointSize[0.015], QPoint2[0, "0", { $\delta$ , - $\delta$ ]},
    {Red, QPoint2[P, "P", {- $\delta$ ,  $\delta$ ]},
    {Blue, QPoint2[Q1, "Q1", -1.5 {0,  $\delta$ ]},
    {Green, QPoint2[Q2, "Q2", -1.5 {0,  $\delta$ ]}]}];
  (*Print[points];*)
  lines = {InfiniteLine[{0, {1, Tan[ $\theta$ Kelvin]}]},
    {Blue, Arrowheads[Small], {Directive[Dashed], InfiniteLine[P, {1, mCurve1}]},
    Arrow[{Q1, P}]}, {Green, Arrowheads[Medium],
    {Directive[Dashed], InfiniteLine[P, {1, mCurve2}]}, Arrow[{Q2, P}]}];
  directives = {Axes  $\rightarrow$  None, ImageSize  $\rightarrow$  {500, 300}, PlotRange  $\rightarrow$  {xrange, yRange}};
  G[1] = Graphics[{axes, points, lines}, Evaluate@directives];
  G[2] =
    ParametricPlot[{ $\theta$ Curve[ $\alpha$ , a2],  $\theta$ Curve[ $\alpha$ , a3]},
      { $\alpha$ , 0,  $\pi/2$ }, PlotStyle  $\rightarrow$  {Blue, Green}, Evaluate@directives];
  Show[G[2], G[1]]

```

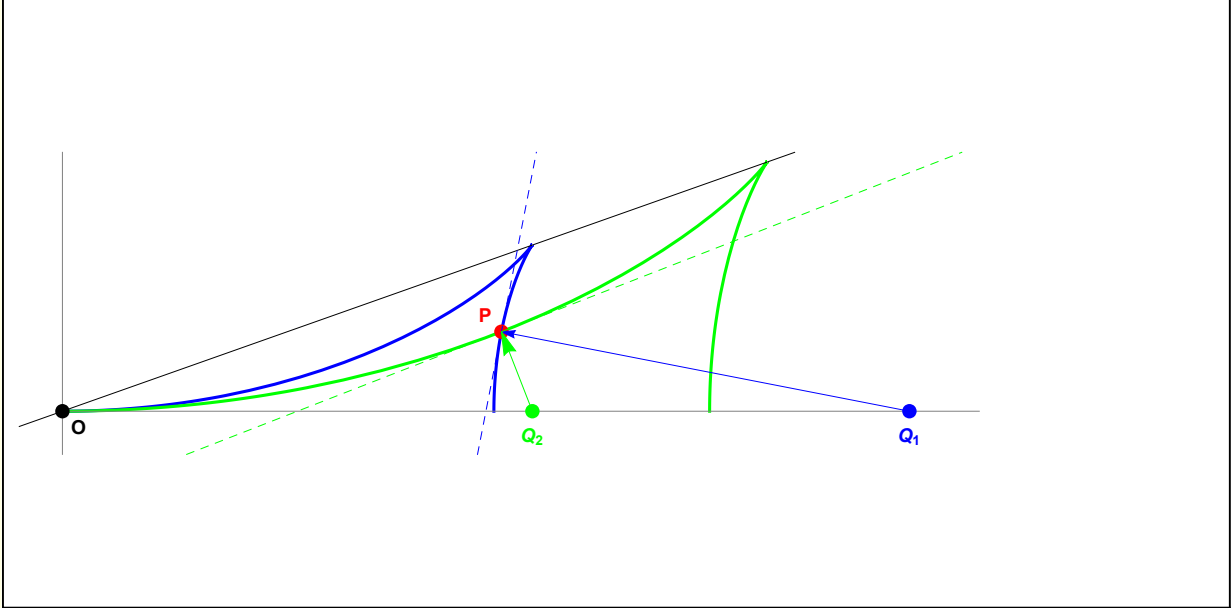


Figure A1

