Mechanics - Morin Type 2 Problems 9.10-13 10-30-18

N. T. Gladd

Initialization: Be sure the file *NTGUtilityFunctions.m* is in the same directory as that from which this notebook was loaded. Then execute the cell immediately below by mousing left on the cell bar to the right of that cell and then typing "shift" + "enter". Respond "Yes" in response to the query to evaluate initialization cells.

```
SetDirectory[NotebookDirectory[]];
 (* set directory where source files are located *)
Get["NTGUtilityFunctions.m"]; (* Load utilities package *)
```

Background

This is the fourth in a series of Mathematica notebooks on classical mechanics. This series was motivated by a close reading and problem solving project I undertook in 2014. The focus of my attention was the text **Introduction to Classical Mechanics with Problems and Solutions**, by David Morin. This is a good book from which to learn and has a great collection of problems. I purchased it and recommend that those with interests in this topic acquire it for their library. I do note that an earlier version can be found on the web. This year, when I returned to this project, I decided to focus on generating Mathematica notebooks on material covered in *Chapter 9 Angular Momentum, Part II (General $\vec{L}$),* which deals the 3-D rigid body dynamics. This topic is notorious difficult/confusing for students and I felt I just skimmed by as a graduate student. I return in retirement after all these years to pay my dues and really understand how to solve problems is this area.

Although Morin's Chapter 9 guides these notebooks, I made frequent use of other sources such as textbooks available in libraries or on the web. I also found lots of video lectures available on YouTube. Confused about some physics topics? Google it and you'll be amazed what you find. Some relevant texts are

Classical Mechanics, Hebert Goldstein (my original text at University, late 60s). Newer versions exist.
Mechanics: Volume1 A Course in Theoretical Physics, L. D. Landau and E. M. Lifshitz.
Classical Mechanics, John. R. Taylor
Classical Mechanics of Particles and Systems, Stephen T. Thornton, Jerry B. Marion
Analytical Mechanics, G. R. Fowles, G. L. Cassiday

Analytical Mechanics, Louis N. Hand, Janet D. Finch

I find Mathematica useful for this topic. It facilitates calculations, provides a vehicle for creating instructive visualizations and allows one to quickly generate numerical solutions. Mathematica is a favorite tool of mine but I think it is crucially important to also work with pen and paper. Our brains are closely linked to our hands and one thinks differently with a pen in hand than when sitting before a computer screen. For serious thoughts on this, read *The Craftsman*, by Richard Sennett.

Purpose

I solve several of Morin's "type 2" problems - What is the frequency of rotation of a rigid body to which a torque is applied?

The specific problems are

1 Morin 9.10 Rotating stick constrained by strings

2 Morin 9.11 Constrained stick driven by rotating ring

3 Morin 9.12 Three approaches to a spherical pendulum

4 Morin 9.13 Ring rolling in a cone

1 Morin 9.10 Rotating stick constrained by strings

Morin 9.10 is a Type 2 problem as described in Section 9.4.2.
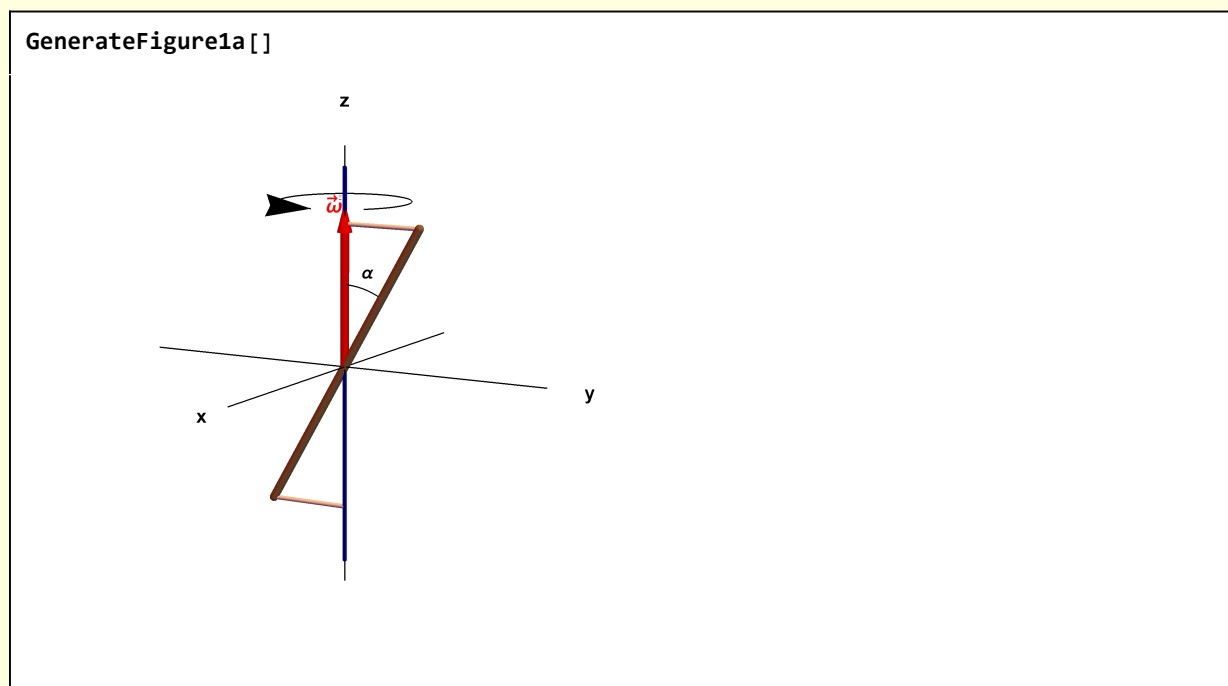
The problem set up is

```
GenerateFigure1a[]
```

Figure 1a Stick rotating about z-axis while restrained by strings.

I follow Morin's solution procedure for Type 2 problems, for which the angular frequency is sought. In this case, however, the angular frequency is specified and it is the tension in the constraining strings to be calculated.

For such problems, the calculation is relatively simple, but getting all of the directions of the various vectors can be confusing. I use Mathematica to profusely visualize the various steps.

Step 1: Calculate the principal moments.

I choose the x_3-principal axis to lie along the stick. Operationally, e_3 is obtained by rotating e_z about the e_x axis by $-\alpha$ (clockwise rotation is negative).
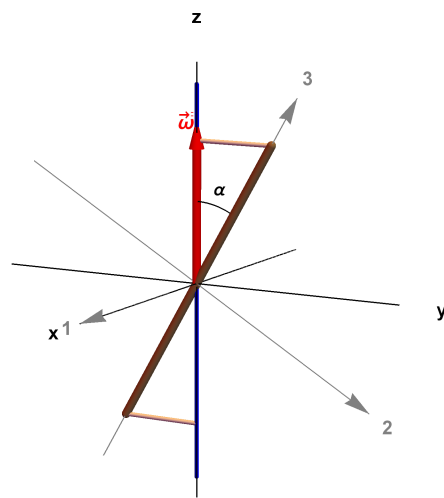
```
GenerateFigure1b[]
```


Figure 1b Choice of principal axes

The inertia tensor expressed in principal component coordinates is

```
I = {{x₂² + x₃², -x₁ x₂, -x₁ x₃},
     {-x₂ x₁, x₁² + x₃², -x₂ x₃},
     {-x₃ x₁, -x₃ x₂, x₁² + x₂²}};
```

```
w1[1] = With[{ρ = M/ℓ},
   Integrate[ρ I /. {x₁ → 0, x₂ → 0}, {x₃, -ℓ/2, ℓ/2}] ]
```

$$\left\{\left\{\frac{M\,\ell^2}{12}, 0, 0\right\}, \left\{0, \frac{M\,\ell^2}{12}, 0\right\}, \{0, 0, 0\}\right\}$$

```
(w1[1] // MatrixForm) // TraditionalForm
```

$$\begin{pmatrix} \frac{M \ell^2}{12} & 0 & 0 \\ 0 & \frac{M \ell^2}{12} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Step 2: Find $\vec{L}$.
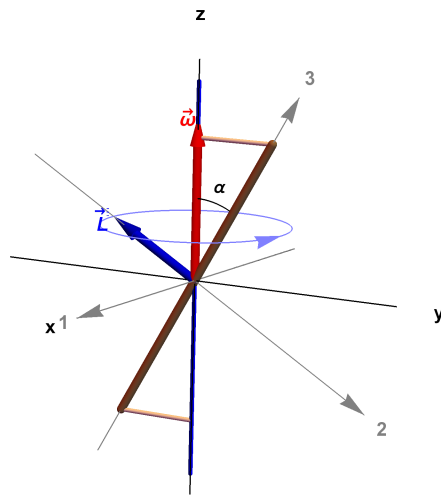
```
GenerateFigure1c[]
```


Figure 1c Angular momentum

$$\vec{L} = \overset{\leftrightarrow}{I} \cdot \vec{\omega}$$

```
w1[2] = {L1, L2, L3} == Dot[w1[1], {ω1, ω2, ω3}]
```

$$\{L1, L2, L3\} == \left\{ \frac{1}{12} M \ell^2 \omega1, \frac{1}{12} M \ell^2 \omega2, 0 \right\}$$

where the components of ω are along the principal axes.

Step 3: Find $d\vec{L}/dt$.

From Figure 3, $\omega1 = 0$, $\omega2 = -\omega \sin(\alpha)$, $\omega3 = \omega \cos(\alpha)$

```
w1[3] = w1[2] /. {ω1 → 0, ω2 → -ω Sin[α], ω3 → ω Cos[α]}
```

$$\{L1, L2, L3\} == \left\{ 0, -\frac{1}{12} M \ell^2 \omega \sin[\alpha], 0 \right\}$$

The magnitude of $\vec{L}$ is constant but the direction changes and the stick rotates about the vertical axis.

$$v_{\text{tip}} = |L| \cos(\alpha)\,\omega$$

```
w1[4] = dLdt == Simplify[Norm[w1[3]〚2〛] ω Cos[α],
    Assumptions → {M > 0, ℓ > 0, ω > 0, Sin[α] > 0}] /. Sin[2 α] → 2 Sin[α] Cos[α]
```

$$\text{dLdt} == \frac{1}{12} M\,\ell^2\,\omega^2\,\text{Cos}[\alpha]\,\text{Sin}[\alpha]$$

At the time depicted in Figure 1c

```
w1[5] = dLdtVec == dLdt {1, 0, 0} /. (w1[4] /. Equal → Rule)
```

$$\text{dLdtVec} == \left\{ \frac{1}{12} M\,\ell^2\,\omega^2\,\text{Cos}[\alpha]\,\text{Sin}[\alpha], 0, 0 \right\}$$

The constrained rotating stick, with all forces and vectors depicted is
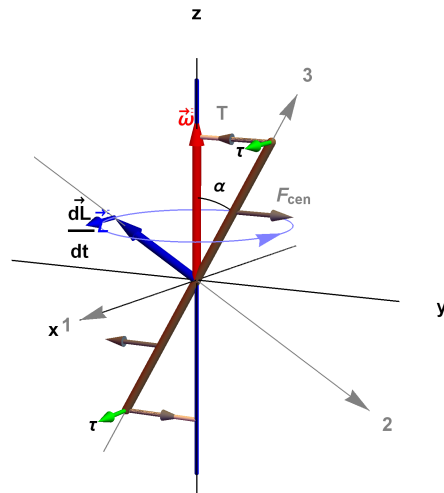
```
GenerateFigure1d[]
```


Figure 1d Constrained rotating stick with all forces shown

Step 3: Calculate the torque.

The tension T in the strings is balanced against the centrifugal force on the stick. Note, that for problems involving a force distributed along the stick, it is appropriate to assume the force is concentrated at the COM. Above the origin, the distance to the COM is ($\ell/2$) along the $\hat{e}_z$ axis. Below the axis, the distance is ($-\ell/2$) along the $\hat{e}_z$ axis. Because the tension in the string is a distance ℓ, it is tempting to assume the moment arm is ℓ — but that is incorrect.

Consistent with Figure 4

$$\vec{\tau} = \vec{r}_{top} \times \vec{T}_{top} + \vec{r}_{bot} \times \vec{T}_{bot} = \frac{\ell}{2}\,\hat{e}_3 \times T\left(-\hat{e}_y\right) + \frac{\ell}{2}\left(-\hat{e}_3\right) \times T\left(\hat{e}_y\right) = -\ell\,T\,\hat{e}_3 \times \hat{e}_y$$

```
w1[6] = {τx, τy, τz} ==
   Module[{ex = {1, 0, 0}, ey = {0, 1, 0}, ez = {0, 0, 1}, e3},
    e3 = RotationMatrix[-α, ex].ez;
    -ℓ T Cross[e3, ey] ]
```

$\{\tau_x, \tau_y, \tau_z\} == \{T \ell \cos[\alpha], 0, 0\}$

Step 4: Equate the torque $\vec{\tau}$ with $\vec{dL/dt}$

```
w1[7] = w1[5]〚2, 1〛 == w1[6]〚2, 1〛
```

$\dfrac{1}{12} M \ell^2 \omega^2 \cos[\alpha] \sin[\alpha] == T \ell \cos[\alpha]$

```
w1[8] = Solve[w1[7], T]〚1, 1〛
```

$T \rightarrow \dfrac{1}{12} M \ell \omega^2 \sin[\alpha]$

which is Morin's result (9.123)

2 Morin 9.11 Constrained stick driven by rotating ring

Morin 9.11 is a Type 2 problem as described in Section 9.4.2.

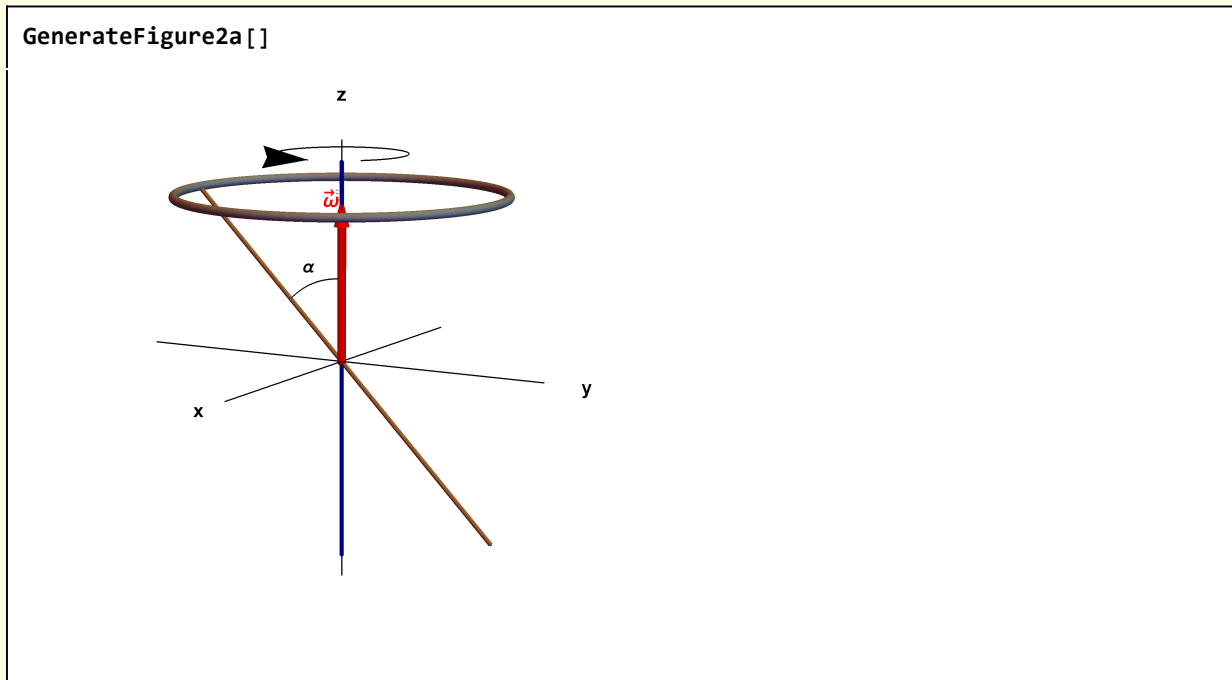The problem set up is

```
GenerateFigure2a[]
```


Figure 2a Constrained stick driven by rotating ring.

I follow Morin's solution procedure for Type 2 problems, for which the angular frequency is sought.
In this case, however, the angular frequency is specified and the tension in the constraining strings is to be calculated.

Step 1: Calculate the principal moments.
I choose the x_3-principal axis to lie along the stick. Operationally, e_3 is obtained by rotating e_z about the e_x axis by α.
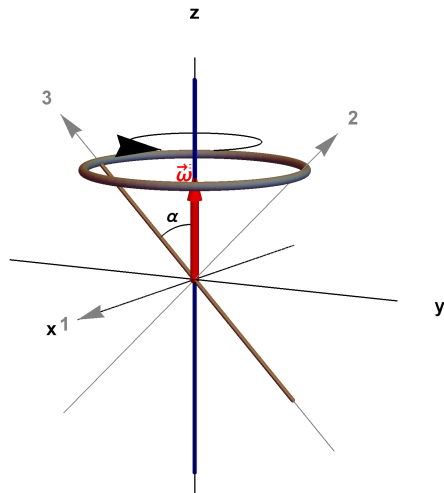
```
GenerateFigure2b[]
```


Figure 2b Principal axes

The inertia tensor expressed in principal component coordinates is

```
I = {{x₂² + x₃², -x₁ x₂, -x₁ x₃},
     {-x₂ x₁, x₁² + x₃², -x₂ x₃},
     {-x₃ x₁, -x₃ x₂, x₁² + x₂²}};
```

```
w2[1] = With[{ρ = M/ℓ},
   Integrate[ρ I /. {x₁ → 0, x₂ → 0}, {x₃, -ℓ/2, ℓ/2}]]
```

$$\left\{\left\{\frac{M\ell^2}{12}, 0, 0\right\}, \left\{0, \frac{M\ell^2}{12}, 0\right\}, \{0, 0, 0\}\right\}$$

```
(w2[1] // MatrixForm) // TraditionalForm
```

$$\begin{pmatrix} \frac{M\ell^2}{12} & 0 & 0 \\ 0 & \frac{M\ell^2}{12} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
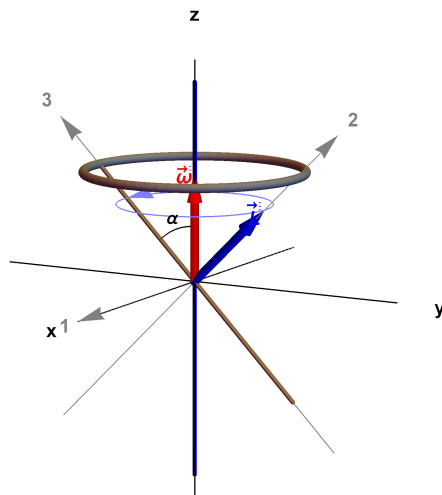
Step 2: Find $\vec{L}$.

```
GenerateFigure2c[]
```


Figure 2c Angular momentum

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$$

```
w2[2] = {L1, L2, L3} == Dot[w2[1], {ω1, ω2, ω3}]
```

$$\{L1, L2, L3\} == \left\{ \frac{1}{12} M \ell^2 \omega 1, \frac{1}{12} M \ell^2 \omega 2, 0 \right\}$$

where the components of ω are along the principal axes.

Step 3: Find d$\vec{L}$/dt.

From Figure 2c, $\omega 1 = 0$, $\omega 2 = \omega \sin(\alpha)$, $\omega 3 = \omega \cos(\alpha)$

```
w2[3] = w2[2] /. {ω1 → 0, ω2 → ω Sin[α], ω3 → ω Cos[α]}
```

$$\{L1, L2, L3\} == \left\{ 0, \frac{1}{12} M \ell^2 \omega \, \text{Sin}[\alpha], 0 \right\}$$

The magnitude of $\vec{L}$ is constant but the direction changes and the stick rotates about the vertical axis.

$$v_{\text{tip}} = |L| \cos(\alpha) \, \omega$$

```
w2[4] = dLdt == Simplify[Norm[w2[3][[2]] ] ω Cos[α],
       Assumptions → {M > 0, ℓ > 0, ω > 0, Sin[α] > 0}] /.
    Sin[2 α] → 2 Sin[α] Cos[α] /. Abs[x_]² → x²
```

$$dLdt == \frac{1}{12} M \ell^2 \omega^2 \cos[\alpha] \sin[\alpha]$$

At the time depicted in Figure 2c

```
w2[5] = dLdtVec == dLdt {-1, 0, 0} /. (w2[4] /. Equal → Rule) // Simplify
```

$$dLdtVec == \left\{ -\frac{1}{12} M \ell^2 \omega^2 \cos[\alpha] \sin[\alpha], 0, 0 \right\}$$

The constrained rotating stick, with all forces and vectors depicted is
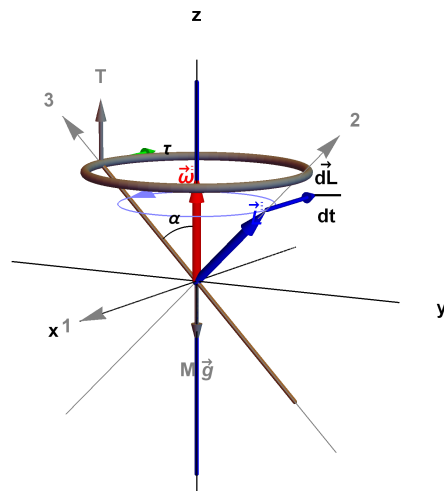
```
GenerateFigure2d[]
```


Figure 2d Constrained rotating stick with all forces shown

Step 3: Calculate the torque.

The tension T in the strings is balanced against the centrifugal force on the stick. Note, that for problems involving a force distributed along the stick, it is appropriate to assume the force is concentrated at the COM. Above the origin, the distance to the COM is ($\ell/2$) along the $\hat{e}_z$ axis. Below the axis, the distance is ($-\ell/2$) along the $\hat{e}_z$ axis. Because the tension in the string is a distance ℓ, it is tempting to assume the moment arm is ℓ — but that is incorrect.

Consistent with Figure 2d

$$\vec{\tau} = \vec{r} \times \vec{T} = (\ell/2)\,\hat{e}_3 \times T\,\hat{e}_z$$

```
w2[6] = {τ_x, τ_y, τ_z} ==
   Module[{ex = {1, 0, 0}, ey = {0, 1, 0}, ez = {0, 0, 1}, e3},
    e3 = RotationMatrix[α, ex].ez;
    ℓT
    ── Cross[e3, ez] ]
    2
```

$$\{\tau_x,\ \tau_y,\ \tau_z\} == \left\{ -\frac{1}{2}\,T\,\ell\,\mathrm{Sin}[\alpha],\ 0,\ 0 \right\}$$

Step 4: Equate the torque $\vec{\tau}$ with $\vec{dL}/dt$

```
w2[7] = w2[6]〚2, 1〛 == w2[5]〚2, 1〛
```

$$-\frac{1}{2}\,T\,\ell\,\mathrm{Sin}[\alpha] == -\frac{1}{12}\,M\,\ell^2\,\omega^2\,\mathrm{Cos}[\alpha]\,\mathrm{Sin}[\alpha]$$

```
w2[8] = Solve[w2[7], ω]〚2, 1〛
```

$$\omega \rightarrow \frac{\sqrt{6}\ \sqrt{T}\ \sqrt{\mathrm{Sec}[\alpha]}}{\sqrt{M}\ \sqrt{\ell}}$$

which is Morin's result (9.125)

3 Morin 9.12 Three approaches to a spherical pendulum

Approach A: For a classical $\vec{F} = m\,\vec{a}$ approach, the problem set up is
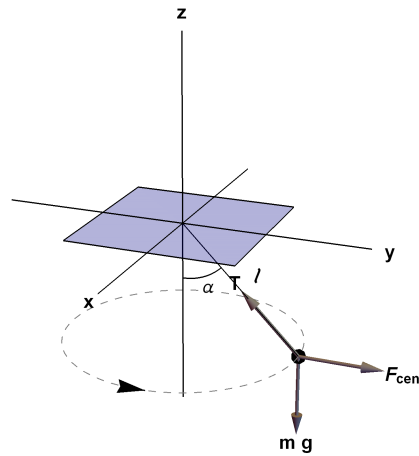
```
GenerateFigure31[]
```

Spherical pendulum
$\vec{F} = m \, \vec{a}$ method

Figure 3a Forces on spherical pendulum

This is just an exercise is balancing forces

```
w3[1] = T Cos[α] == m g
```

$T \, Cos[\alpha] \, == \, g \, m$

and perpendicular tension against centrifugal force

```
w3[2] = T Sin[α] == m v²/r
```

$T \, Sin[\alpha] \, == \, \dfrac{m \, v^2}{r}$

```
w3[3] = w3[2] /. v → ω r /. r → ℓ Sin[α]
```

$T \, Sin[\alpha] \, == \, m \, \ell \, \omega^2 \, Sin[\alpha]$

```
w3[4] = w3[3] /. Solve[w3[1], T]〚1〛
```

$$g\,m\,Tan[\alpha] \,=\!= m\,\ell\,\omega^2\,Sin[\alpha]$$

```
w3[5] = Solve[w3[4], ω]〚2, 1〛
```

$$\omega \to \frac{\sqrt{g}\ \sqrt{Sec[\alpha]}}{\sqrt{\ell}}$$

which is the classical result.

Approach B: For $\vec{\tau} = d\,\vec{L}/dt$ approach, with the reference point for angular momentum at the pivot point.

```
GenerateFigure3b[]
```


Spherical pendulum
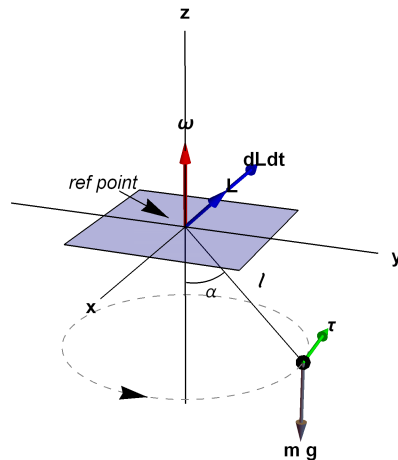$d\vec{L}/dt = \vec{\tau}$ about pivot point method

Figure 3b Forces and angular momentum vectors for spherical pendulum relative to pivot point

Using the pivot point as the reference point, the angular moment of the pendulum is

$$\vec{L} = \vec{r} \times m\,\vec{v} = m\,\vec{r} \times \left(\vec{\omega} \times \vec{r}\right)$$

```
w3B[1] = {Lx, Ly, Lz} == With[{r = {0, ℓ Sin[α], -ℓ Cos[α]}, w = {0, 0, ω}},
    m Cross[r, Cross[w, r]]] // Simplify
```

$$\{Lx, Ly, Lz\} == \{0, m \ell^2 \omega \cos[\alpha] \sin[\alpha], m \ell^2 \omega \sin[\alpha]^2\}$$

Note

```
w3B[2] = LMag == Norm[w3B[1]〚2〛] /. Abs[x_] → x // Simplify;
w3B[2] = w3B[2] // PowerExpand
```

$$m \ell^2 \omega \sin[\alpha] == \text{LMag}$$

The tip of L is moving in a circle of size L $\omega \cos(\alpha)$. Consistent with the picture, $\vec{L}$ has direction $-e_x$

```
w3B[3] = dLdt == Cos[α] ω LMag {-1, 0, 0} /. Solve[w3B[2], LMag]〚1〛
```

$$\text{dLdt} == \{-m \ell^2 \omega^2 \cos[\alpha] \sin[\alpha], 0, 0\}$$

The torque about the pivot point is

$$\vec{\tau} = \vec{r} \times m \vec{g}$$

```
w3B[4] = {τx, τy, τz} ==
    With[{r = {0, ℓ Sin[α], -ℓ Cos[α]}, ez = {0, 0, 1}},
    m g Cross[r, -ez]]
```

$$\{\tau x, \tau y, \tau z\} == \{-g m \ell \sin[\alpha], 0, 0\}$$

Equation dLxdt with τx

```
w3B[5] = w3B[3]〚2, 1〛 == w3B[4]〚2, 1〛
```

$$-m \ell^2 \omega^2 \cos[\alpha] \sin[\alpha] == -g m \ell \sin[\alpha]$$

```
w3B[6] = Solve[w3B[5], ω]〚2, 1〛
```

$$\omega \to \frac{\sqrt{g} \ \sqrt{\text{Sec}[\alpha]}}{\sqrt{\ell}}$$

as expected.

Approach C: $\vec{\tau} = d \vec{L}/dt$ approach, with the reference point at the point mass.

```
GenerateFigure3c[]
```

Spherical pendulum
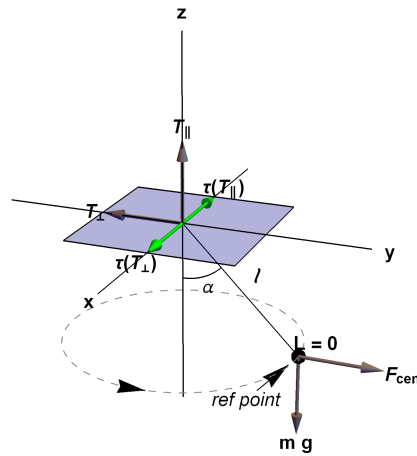
$d\vec{L}/dt = \vec{\tau}$ **about mass method**

Figure 3c Forces and angular momentum vectors for spherical pendulum relative to COM

With the point mass as origin, $\vec{L} = 0$. Thus, $\vec{\tau} = 0$. This means that the torques from the tension in the support must vanish.

$$\vec{\tau} = 0 = \vec{r} \times \vec{T} = \vec{r} \times T_{\perp}(-\hat{e}_y) + T_{\parallel} \vec{r} \times \hat{e}_z$$

The first component is in the $\hat{e}_x$ direction

```
w3C[1] = {τTperx, τTpery, τTperz} == Tper ℓ Cross[{0, -Sin[α], Cos[α]}, {0, -1, 0}]

{τTperx, τTpery, τTperz} == {Tper ℓ Cos[α], 0, 0}
```

The second component is in the $-\hat{e}_x$ direction

```
w3C[2] = {τTpelx, τTpely, τTpelz} == Tpel ℓ Cross[{0, -Sin[α], Cos[α]}, {0, 0, 1}]

{τTpelx, τTpely, τTpelz} == {-Tpel ℓ Sin[α], 0, 0}
```

So,

```
w3C[3] = w3C[1][[2, 1]] + w3C[2][[2, 1]] == 0
```

$Tper \, \ell \, Cos[\alpha] - Tpel \, \ell \, Sin[\alpha] == 0$

```
w3C[4] = w3C[3] /. Tper → m ω² ℓ Sin[α] /. Tpel → m g
```

$- g \, m \, \ell \, Sin[\alpha] + m \, \ell^2 \, \omega^2 \, Cos[\alpha] \, Sin[\alpha] == 0$

```
w3C[5] = Solve[w3C[4], ω][[2, 1]]
```

$$\omega \to \frac{\sqrt{g} \ \sqrt{Sec[\alpha]}}{\sqrt{\ell}}$$

as before.

4 Morin 9.13 Ring rolling in a cone

Morin 9.13 is a Type 2 problem as described in Section 9.4.2.
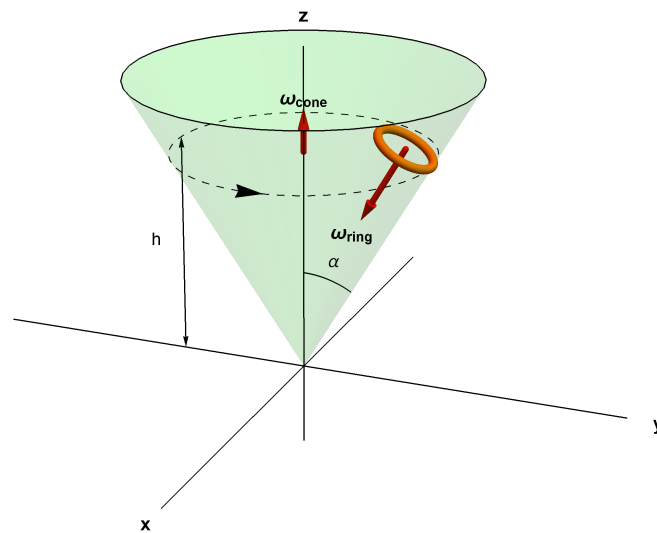
```
GenerateFigure4a[]
```

Ring rolling in cone

Figure 4a Geometry of the problem

A natural origin for the problem is the center of mass of the ring. I illustrate how the principal axes are oriented in the following 2-D perspective.
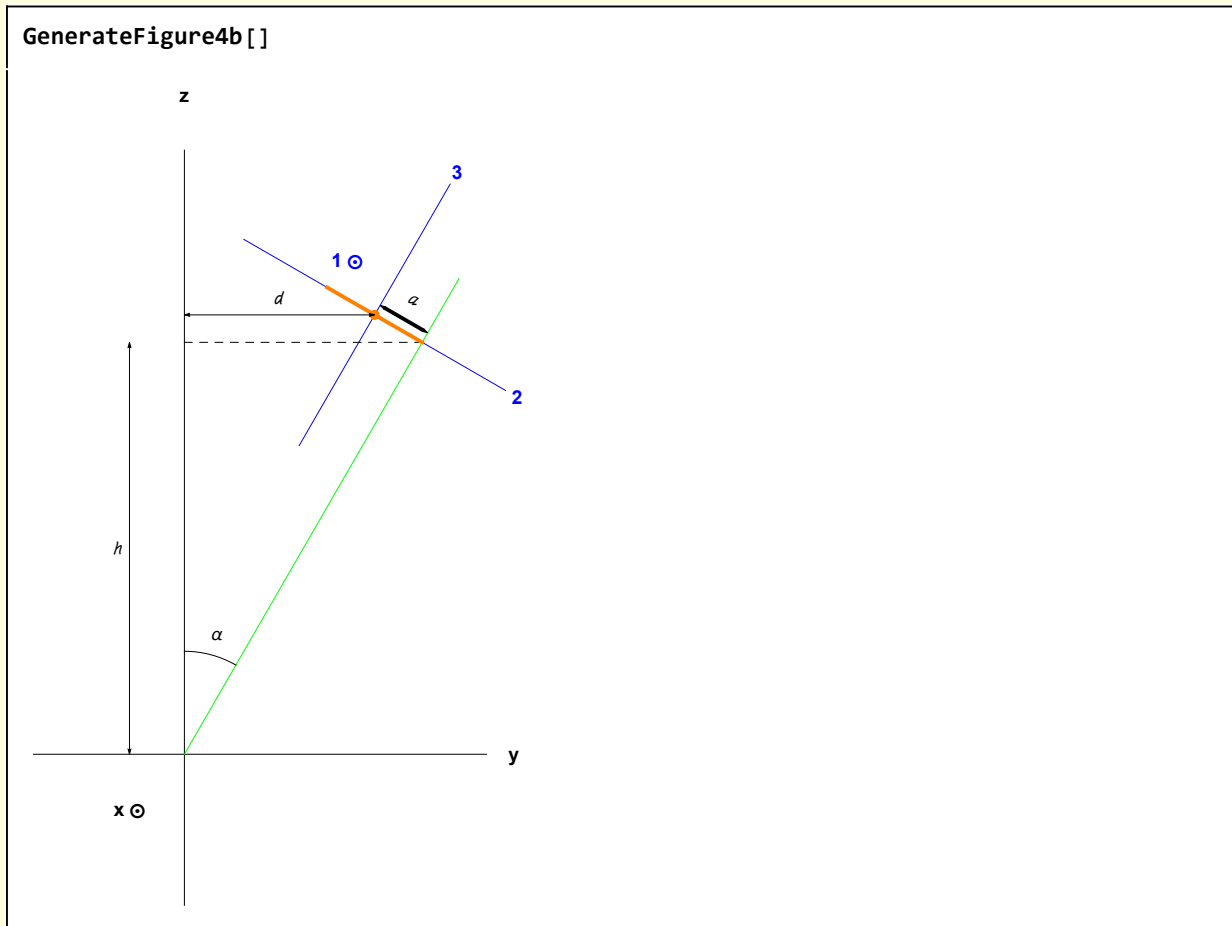
```
GenerateFigure4b[]
```


Figure 4b A 2-D slice of Figure 4a showing the principal axes

With two axes of rotation, the angular momentum is most easily calculated from the differential expression

$$d\vec{L} = dm\,\vec{r} \times \vec{v} = dm\,\vec{r} \times \left(\vec{\omega} \times \vec{r}\right) = dm\,\vec{r} \times \left(\vec{\omega}_c \times \vec{r}\right) + dm\,\vec{r} \times \left(\vec{\omega}_r \times \vec{r}\right)$$

where dm represents an element of mass on the ring where $\vec{r}$ denotes a position vector from the COM of the ring to the position of a mass element on the ring, and $\vec{\omega}_c$ and $\vec{\omega}_r$ are the angular velocities of the cone and the ring.

For the instant (say t = 0) depicted in Figures 1 and 2, and for dm chosen to have it's rightmost extent, the expression for $\vec{r}$ in terms of Cartesian coordinates relative the the COM of the rigng

$$\vec{r} = a\,\hat{e}_2 = a\,\{0,\ \cos(\alpha),\ -\sin(\alpha)\}$$
$$\vec{\omega}_c = \omega_c\,\{0,\ 0,\ 1\}$$
$$\vec{\omega}_r = -\omega_c\,\{0,\ \sin(\alpha),\ \cos(\alpha)\}$$

```
w4[1] = dL == Module[{r = a {0, Cos[α], -Sin[α]},
     wr = -ωr {0, Sin[α], Cos[α]}, wc = ωc {0, 0, 1}},
    dm Cross[r, Cross[wc, r]] + dm Cross[r, Cross[wr, r]]] // Simplify
```

$$dL == \left\{0, \, dm \, a^2 \left(-\omega r + \omega c \, Cos[\alpha]\right) Sin[\alpha], \, dm \, a^2 \, Cos[\alpha] \left(-\omega r + \omega c \, Cos[\alpha]\right)\right\}$$

where I use a double stroke notation to indicate vector quantities.

These quantities do not have an explicit dependence on the angle around the ring, so

$$\int dm = \int_0^{2\pi} \left(\frac{M}{2\pi a}\right) a \, d\beta = M$$

Thus

```
w4[2] = L == w4[1]〚2〛 /. dm → M
```

$$L == \left\{0, \, M \, a^2 \left(-\omega r + \omega c \, Cos[\alpha]\right) Sin[\alpha], \, M \, a^2 \, Cos[\alpha] \left(-\omega r + \omega c \, Cos[\alpha]\right)\right\}$$

Note that there is a no-slip kinematical relationship between ω_r and ω_c

$$h \tan(\alpha) \, d\phi_c = a \, d\phi_r \text{ implies}$$

$$h \tan(\alpha) \, \omega_c = a \, \omega_r$$

The problem is to be solved in the limit $a \ll h \tan(\alpha)$, so $\omega_c \ll \omega_r$.
Thus

```
w4[3] = w4[2] /. ωc → 0
```

$$L == \left\{0, \, -M \, a^2 \, \omega r \, Sin[\alpha], \, -M \, a^2 \, \omega r \, Cos[\alpha]\right\}$$

This approximation implies only ω_{ring} in Figure 1 contributes to the angular momentum

Consistent with Figure 4a corresponding to t = 0, $\vec{L}(t)$ is

```
w4[4] = L[t] == -M a² ωr {-Sin[ωc t] Sin[α], Cos[ωc t] Sin[α], Cos[α]}
```

$$L[t] == \left\{M \, a^2 \, \omega r \, Sin[\alpha] \, Sin[t \, \omega c], \, -M \, a^2 \, \omega r \, Cos[t \, \omega c] \, Sin[\alpha], \, -M \, a^2 \, \omega r \, Cos[\alpha]\right\}$$

```
w4[5] = (D[#, t]) & /@ w4[4] // Simplify
```

$$L'[t] == \left\{M \, a^2 \, \omega c \, \omega r \, Cos[t \, \omega c] \, Sin[\alpha], \, M \, a^2 \, \omega c \, \omega r \, Sin[\alpha] \, Sin[t \, \omega c], \, 0\right\}$$

At t = 0

```
w4[6] = w4[5] /. t → 0
```

$\mathbb{L}'[0] == \{M a^2 \omega c \omega r \, \text{Sin}[\alpha], 0, 0\}$

```
GenerateFigure4c[]
```

Showing $\vec{\omega}, \vec{L}$, and $\vec{dL}/dt$ vectors
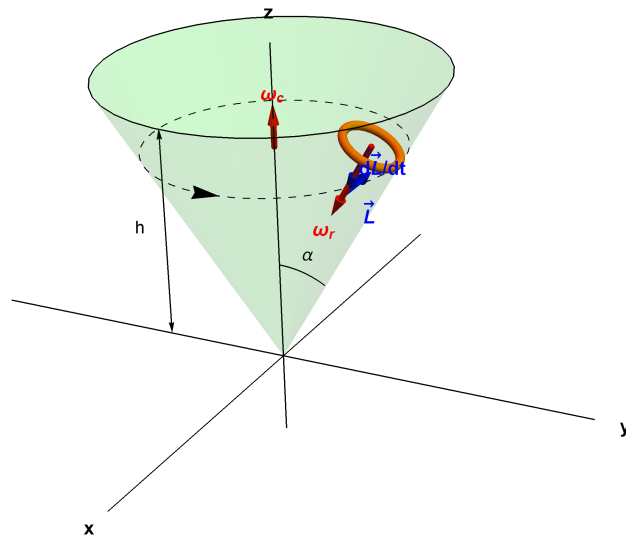

Figure 4c Showing the vectors $\vec{\omega}, \vec{L}$, d $\vec{L}$/dt

To calculate the torque on the ring, consider the forces at the point of contact for the case when the radius of the ring is very small. The force of gravity is downward, there is an outward centrifugal force outward due to ω_c, a normal force because the ring is in contact with the cone, and a force of friction between ring and cone. The centrifugal force associated with the ring spinning about itself can produce
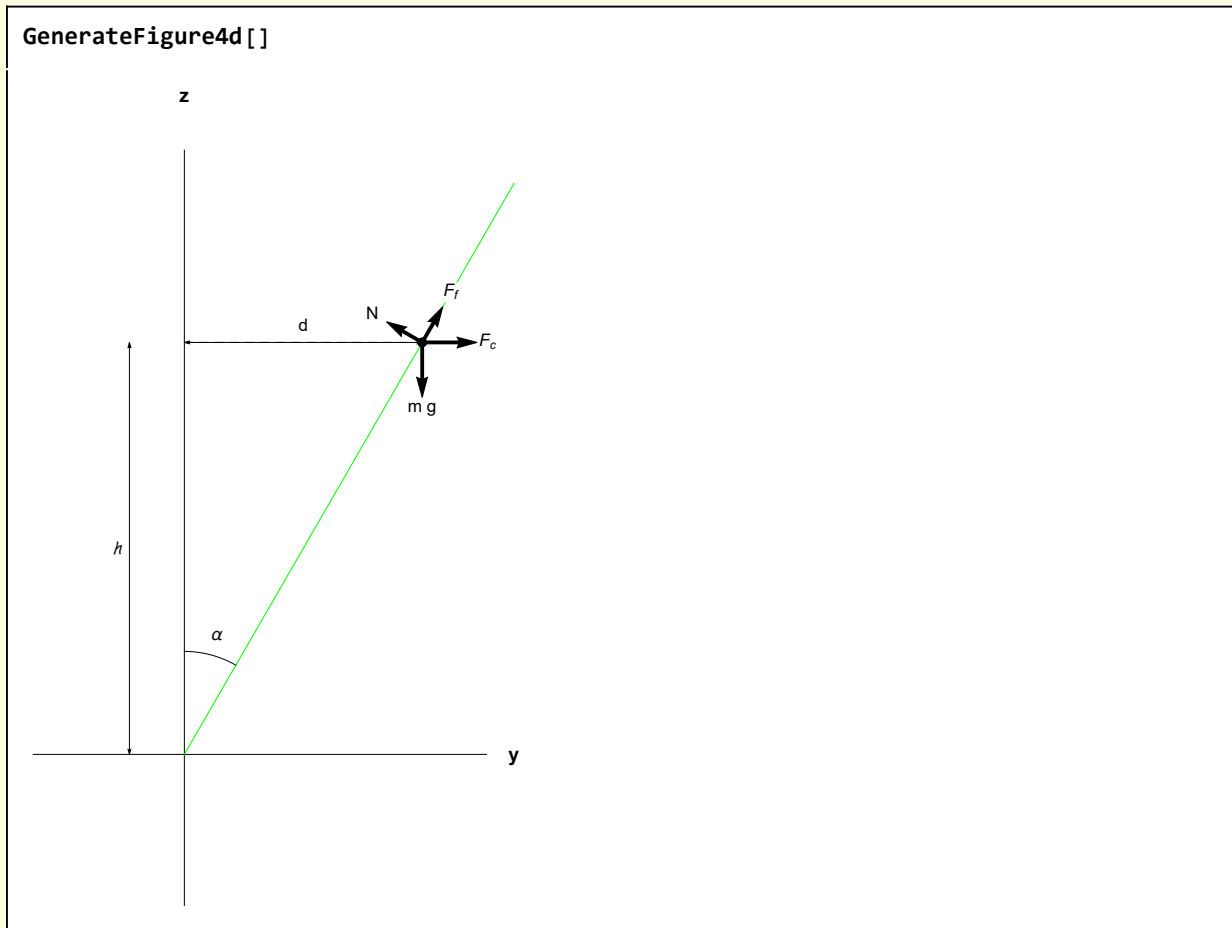
no torque about the COM.

```
GenerateFigure4d[]
```


Figure 4d Cross section showing the various forces

Force balance yields

```
w4[7] = 𝒩 Sin[α] + Ff Cos[α] == M g
```

$Ff \, Cos[\alpha] + \mathcal{N} \, Sin[\alpha] \; == \; g \, M$

```
w4[8] = 𝒩 Cos[α] == Ff Sin[α] + Fc
```

$\mathcal{N} \, Cos[\alpha] \; == \; Fc + Ff \, Sin[\alpha]$

```
w4[9] = (Solve[{w4[7], w4[8]}, {Ff, 𝒩}][[1]] // Simplify) /. Rule → Equal
```

$\{Ff \; == \; g \, M \, Cos[\alpha] - Fc \, Sin[\alpha], \; \mathcal{N} \; == \; Fc \, Cos[\alpha] + g \, M \, Sin[\alpha]\}$

Introduce geometrical factors

```
w4[10] = w4[9] /. Fc → M d ω c² /. d → h Tan[α]
```

$\{Ff == g\,M\,Cos[\alpha] - M\,h\,\omega c^2\,Sin[\alpha]\,Tan[\alpha],\ \mathcal{N} == g\,M\,Sin[\alpha] + M\,h\,\omega c^2\,Sin[\alpha]\}$

The normal force points toward to COM of the ring so only the frictional force applies a torque relative to the COM. The vector form of the force is

```
w4[11] = Ff == w4[10]⟦1, 2⟧ {0, Sin[α], Cos[α]}
```

$Ff ==$
$\{0,\ Sin[\alpha]\,(g\,M\,Cos[\alpha] - M\,h\,\omega c^2\,Sin[\alpha]\,Tan[\alpha]),\ Cos[\alpha]\,(g\,M\,Cos[\alpha] - M\,h\,\omega c^2\,Sin[\alpha]\,Tan[\alpha])\}$

The torque is

$$\vec{\tau} = \vec{r} \times \vec{F}_f$$

```
w4[13] =
  t == Module[{r = a {0, Cos[α], -Sin[α]}, F},
     F = Ff /. (w4[11] /. Equal → Rule);
     TEST = {r, F};
     Cross[r, F]] // Simplify
```

$t == \{\frac{1}{2}\,M\,a\,(g - h\,\omega c^2 + (g + h\,\omega c^2)\,Cos[2\,\alpha])\,Sec[\alpha],\ 0,\ 0\}$

where I use t to denote $\vec{\tau}$

Finally,

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$

Recall the expression for $\overrightarrow{dL}/dt$

```
w4[6]
```

$L'[0] == \{M\,a^2\,\omega c\,\omega r\,Sin[\alpha],\ 0,\ 0\}$

Then

```
w4[14] = w4[6]⟦2, 1⟧ == w4[13]⟦2, 1⟧
```

$M\,a^2\,\omega c\,\omega r\,Sin[\alpha] == \frac{1}{2}\,M\,a\,(g - h\,\omega c^2 + (g + h\,\omega c^2)\,Cos[2\,\alpha])\,Sec[\alpha]$

Imposing the kinematical constraint

```
w4[15] = w4[14] /. ωr → d/a ωc /. d → h Tan[α]
```

$$M\,a\,h\,\omega\text{c}^2\,\text{Sin}[\alpha]\,\text{Tan}[\alpha] \;==\; \frac{1}{2}\,M\,a\,\left(g - h\,\omega\text{c}^2 + \left(g + h\,\omega\text{c}^2\right)\,\text{Cos}[2\,\alpha]\right)\,\text{Sec}[\alpha]$$

```
w4[16] = Solve[w4[15], ωc] // Simplify[#, Assumptions → {0 < α < π/2}] &
```

$$\left\{\left\{\omega\text{c} \to -\frac{\sqrt{g}\,\text{Cot}[\alpha]}{\sqrt{2}\,\sqrt{h}}\right\}, \left\{\omega\text{c} \to \frac{\sqrt{g}\,\text{Cot}[\alpha]}{\sqrt{2}\,\sqrt{h}}\right\}\right\}$$

The physical solution is positive

```
w4[17] = w4[16]〚2, 1〛
```

$$\omega\text{c} \to \frac{\sqrt{g}\,\text{Cot}[\alpha]}{\sqrt{2}\,\sqrt{h}}$$

which is Morin's result.

Visualizations

Section 1

```
Clear[GenerateFigure1a];
GenerateFigure1a[] :=
  Module[{α = π / 6, ℓ = 1.5, scale = 1.1, δ = 0.1, vp = {4.5, 2.5, 1}, O, ex, ey,
    ez, axes, range, P, V, axisOfRotation , ωVec, stick, strings, markers, G},
   range = {{- 1, 1}, {- 1, 1}, {- 1,  1}};
   {O, ex, ey, ez} = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
   axes =
    {Black, Line[scale {-ex, ex}], Line[scale { -ey, ey}], Line[scale {- ez, ez}],
     Tex["x", (scale + δ) ex], Tex["y", (scale + δ) ey], Tex["z", (scale + δ) ez]};

   {P, V} = With[{θ = α, φ = π / 2}, {StoC[ℓ, θ, φ], {0, 0, ℓ Cos[θ]}}];
   axisOfRotation = {Blue, Tube[{-0.9 ez, 0.9 ez}, 0.01]};
   ωVec = {Red, Arrow@Tube[{O, {0, 0, ℓ / 2} }, 0.02],
     Tex["ω⃗", O + {0, 0,  ℓ / 2} + {δ, 0, 0}]};
   stick =
    {Brown, Tube[{-P / 2, P / 2}, 0.02]};
   strings = {Directive[Lighter[Yellow, 0.9]],
     Tube[{V / 2, P / 2}, 0.01], Tube[{-V / 2, -P / 2}, 0.01]};

   markers = {
     (* rotation arrow *) {Arrowheads[Medium],
      Arrow@Table[{0.3 Cos[φ], 0.3 Sin[φ], ℓ / 2}, {φ, π / 4, 2 π, 2 π / 64}]},
     (* αArc *) { Line@Table[StoC[ℓ/4, θ, π / 2], {θ, 0, α, α/64}],
       {Black, Text[Stl["α"], StoC[1.2 ℓ/4, α / 2, π / 2]]}}};

   G[1] = Graphics3D[{axes, axisOfRotation, ωVec, stick, strings, markers},
     Boxed → False, ImageSize → 320, ViewPoint → vp, PlotRange → range]];

GenerateFigure1a[]
```
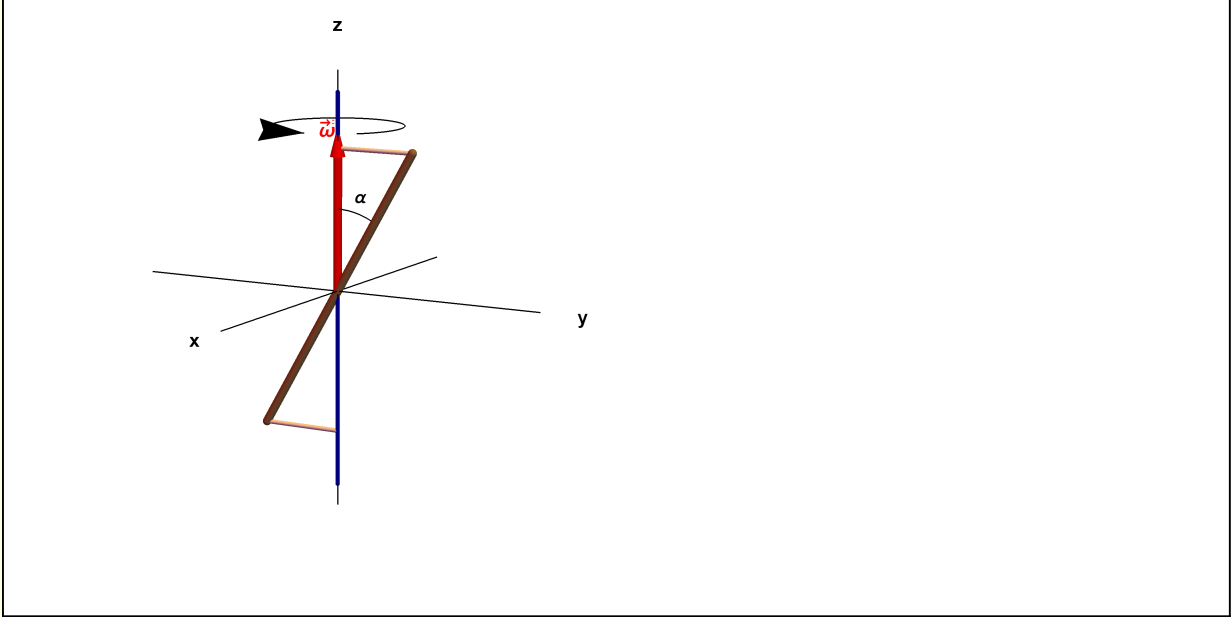
```
Clear[GenerateFigure1b];
GenerateFigure1b[] :=
  Module[{α = π / 6, ℓ = 1.5, scale = 1.1, δ = 0.1, vp = {4.5, 2.5, 1},
    O, ex, ey, ez, e1, e2, e3, axes, axes123, range, P, V, axisOfRotation,
    ωVec, stick, strings, markers, RM = RotationMatrix, G},
   range = {{-1, 1}, {-1, 1}, {-1, 1}};
   {O, ex, ey, ez} = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
   axes =
    {Black, Line[scale {-ex, ex}], Line[scale {-ey, ey}], Line[scale {-ez, ez}],
     Tex["x", (scale + δ) ex], Tex["y", (scale + δ) ey], Tex["z", (scale + δ) ez]};
   {e1, e2, e3} = {ex, RM[-α, e1].ey, RM[-α, e1].ez};
   axes123 = {Gray, Arrowheads[Small], Arrow[{-e1, e1}], Tex["1", 1.1 e1],
     Arrow[{-e2, e2}], Tex["2", 1.1 e2], Arrow[{-e3, e3}], Tex["3", 1.1 e3]};

   {P, V} = With[{θ = α, ϕ = π / 2}, {StoC[ℓ, θ, ϕ], {0, 0, ℓ Cos[θ]}}];
   axisOfRotation = {Blue, Tube[{-0.9 ez, 0.9 ez}, 0.01]};
   ωVec = {Red, Arrow@Tube[{O, {0, 0, ℓ / 2}}, 0.02],
     Tex["ω⃗", O + {0, 0, ℓ / 2} + {δ, 0, 0}]};
   stick =
    {Brown, Tube[{-P / 2, P / 2}, 0.02]};
   strings = {Directive[Lighter[Yellow, 0.9]],
     Tube[{V / 2, P / 2}, 0.01], Tube[{-V / 2, -P / 2}, 0.01]};

   markers = {
      (* αArc *){ Line@Table[StoC[ℓ/4, θ, π / 2], {θ, 0, α, α/64}],

       {Black, Text[Stl["α"], StoC[1.2 ℓ/4, α / 2, π / 2]]}}};

   G[1] = Graphics3D[{axes, axisOfRotation, ωVec, stick, strings, markers, axes123},
     Boxed → False, ImageSize → 320, ViewPoint → vp, PlotRange → range]];

GenerateFigure1b[]
```
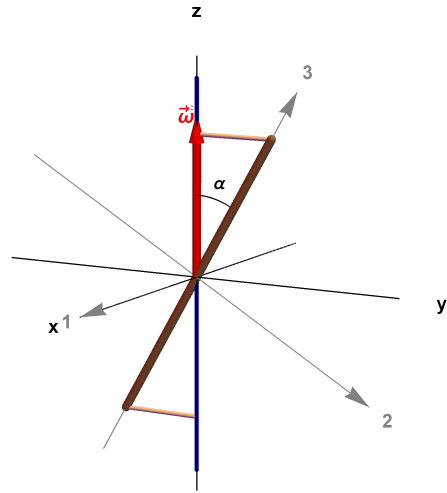
```mathematica
Clear[GenerateFigure1c];
GenerateFigure1c[] :=
  Module[{α = π / 6, ℓ = 1.5, LMag = 0.5, scale = 1.1, δ = 0.1, vp = {4.5, 2.5, 1},
    O, ex, ey, ez, e1, e2, e3, axes, axes123, range, P, V, axisOfRotation,
    ωVec, LVec, LRing, stick, strings, markers, RM = RotationMatrix, G},
   range = {{-1, 1}, {-1, 1}, {-1, 1}};
   {O, ex, ey, ez} = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
   axes =
    {Black, Line[scale {-ex, ex}], Line[scale {-ey, ey}], Line[scale {-ez, ez}],
     Tex["x", (scale + δ) ex], Tex["y", (scale + δ) ey], Tex["z", (scale + δ) ez]};
   {e1, e2, e3} = {ex, RM[-α, e1].ey, RM[-α, e1].ez};
   axes123 = {Gray, Arrowheads[Small], Arrow[{-e1, e1}], Tex["1", 1.1 e1],
     Arrow[{-e2, e2}], Tex["2", 1.1 e2], Arrow[{-e3, e3}], Tex["3", 1.1 e3]};

   {P, V} = With[{θ = α, ϕ = π / 2}, {StoC[ℓ, θ, ϕ], {0, 0, ℓ Cos[θ]}}];
   axisOfRotation = {Blue, Tube[{-0.9 ez, 0.9 ez}, 0.01]};
   ωVec = {Red, Arrow@Tube[{O, {0, 0, ℓ / 2}}, 0.02],
     Tex["ω⃗", O + {0, 0, ℓ / 2} + {δ, 0, 0}]};
   LVec = {Blue, Arrow@Tube[{O, -e2 LMag}, 0.02],
     Tex["L⃗", O + -e2 LMag + {δ, 0, 0}]};
   LRing = {Lighter[Blue, 0.5], Arrowheads[Small],
     Arrow@Table[RM[ϕ, ez].(-e2 LMag), {ϕ, π, 2 π + π, π / 32.}]};

   stick =
    {Brown, Tube[{-P / 2, P / 2}, 0.02]};
   strings = {Directive[Lighter[Yellow, 0.9]],
     Tube[{V / 2, P / 2}, 0.01], Tube[{-V / 2, -P / 2}, 0.01]};

   markers = {
      (* αArc *){ Line@Table[StoC[ℓ/4, θ, π / 2], {θ, 0, α, α/64}],
       {Black, Text[Stl["α"], StoC[1.2 ℓ/4, α / 2, π / 2]]}}};

   G[1] = Graphics3D[
     {axes, axisOfRotation, ωVec, LVec, LRing, stick, strings, markers, axes123},
     Boxed → False, ImageSize → 320, ViewPoint → vp, PlotRange → range]];

GenerateFigure1c[]
```
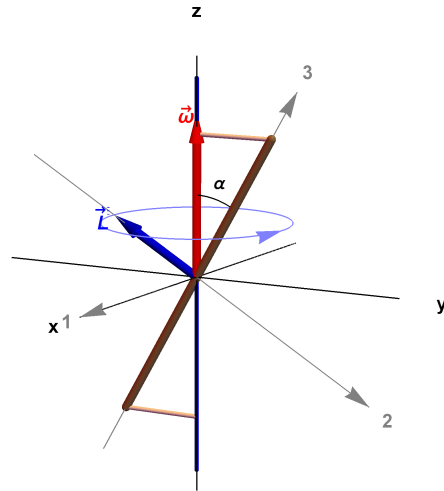
```
Clear[GenerateFigure1d];
GenerateFigure1d[] :=
  Module[{α = π / 6, ℓ = 1.5, LMag = 0.5, dLdtMag = 0.25, τMag = 0.25, scale = 1.1,
    δ = 0.1, vp = {4.5, 2.5, 1}, O, ex, ey, ez, e1, e2, e3, axes, axes123,
    range, P, V, axisOfRotation, ωVec, LVec, LRing, stick, strings, markers,
    FCentrifugalVec, TensionVec, dLdtVec , τVec, RM = RotationMatrix, G},
   range = {{- 1, 1}, {- 1, 1}, {- 1,  1}};
   {O, ex, ey, ez} = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
   axes =
    {Black, Line[scale {-ex, ex}], Line[scale { -ey, ey}], Line[scale {- ez, ez}],
     Tex["x", (scale + δ) ex], Tex["y", (scale + δ) ey], Tex["z", (scale + δ) ez]};
   {e1, e2, e3} = {ex, RM[-α, e1].ey, RM[-α, e1].ez};
   axes123 = {Gray, Arrowheads[Small], Arrow[{-e1, e1}], Tex["1", 1.1 e1],
     Arrow[{-e2, e2}], Tex["2", 1.1 e2], Arrow[{-e3, e3}], Tex["3", 1.1 e3]};

   {P, V} = With[{θ = α, ϕ = π / 2}, {StoC[ℓ, θ, ϕ], {0, 0, ℓ Cos[θ]}}];
   axisOfRotation = {Blue, Tube[{-0.9 ez, 0.9 ez}, 0.01]};
   ωVec = {Red, Arrow@Tube[{O, {0, 0, ℓ / 2} }, 0.02],
     Tex["ω⃗", O + {0, 0, ℓ / 2} + {δ, 0, 0}]};
   LVec = {Blue, Arrow@Tube[{O, -e2 LMag}, 0.02],
     Tex["L⃗", O + -e2 LMag + {δ, 0, 0}]};
   LRing = {Lighter[Blue, 0.5], Arrowheads[Small],
     Arrow@Table[RM[ϕ, ez].(-e2 LMag), {ϕ, π, 2 π + π, π / 32.}]};

   stick =
    {Brown, Tube[{-P / 2, P / 2}, 0.02]};
   strings = {Directive[Lighter[Yellow, 0.9]],
     Tube[{V / 2, P / 2}, 0.01], Tube[{-V / 2, -P / 2}, 0.01]};

   markers = {
```
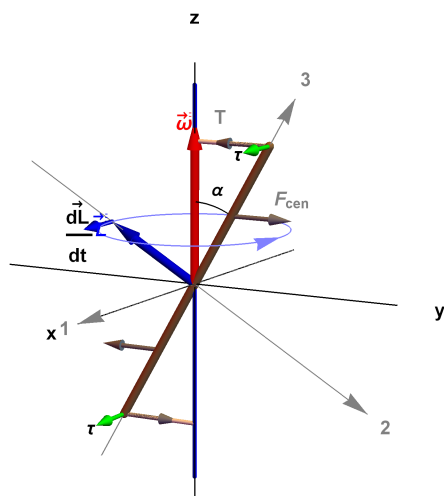
```mathematica
(* αArc *) { Line@Table[StoC[ℓ/4, θ, π/2], {θ, 0, α, α/64}],

   {Black, Text[Stl["α"], StoC[1.2 ℓ/4, α/2, π/2]]}}};


FCentrifugalVec = With[{st = P / 4, fn = P / 4 + {0, 3 δ, 0}},
   {Gray, Vec[{st, fn}, 0.01, 0.03],
    Vec[{-st, -fn}, 0.01, 0.03], Tex["F_cen", fn + {0, 0, δ}]}];
TensionVec = With[{st = P / 2, fn = P / 2 + {0, -3 δ, 0}},
   {Gray, Vec[{st, fn}, 0.01, 0.03],
    Vec[{-st, -fn}, 0.01, 0.03], Tex["T", fn + {-δ, 0, δ}]}];
dLdtVec =
 With[{st = -LMag e2, fn = -LMag e2 + {dLdtMag, 0, 0}},

   {Blue, Vec[{st, fn}, 0.01, 0.03], {Black, Tex["dL⃗/dt", st + 1.2 (fn - st)]}}];

τVec =
 With[{st = ℓ/2 e3, fn = ℓ/2 e3 + {τMag, 0, 0},

   st2 = -ℓ/2 e3, fn2 = -ℓ/2 e3 + {τMag, 0, 0}},
   {Green, Vec[{st, fn}, 0.01, 0.03], {Black, Tex["τ", st + 1.2 (fn - st)]},
    Vec[{st2, fn2}, 0.01, 0.03], {Black, Tex["τ", st2 + 1.2 (fn2 - st2)]}}];


G[1] = Graphics3D[{axes, axisOfRotation, ωVec, LVec, LRing, stick, strings,
    markers, axes123, FCentrifugalVec, TensionVec, dLdtVec, τVec},
   Boxed → False, ImageSize → 320, ViewPoint → vp, PlotRange → range]];

GenerateFigure1d[]
```

Section 2

```
Clear[GenerateFigure2a];
GenerateFigure2a[] :=
  Module[{α = π / 4, rRing = 0.75, scale = 1.1, δ = 0.1, vp = {4.5, 2.5, 1}, ℓ, O, ex,
    ey, ez, axes, range, P, V, axisOfRotation , ωVec, LVec, ring, stick, markers, G},
   range = {{- 1, 1}, {- 1, 1}, {- 1,  1}};
   {O, ex, ey, ez} = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
   axes =
    {Black, Line[scale {-ex, ex}], Line[scale { -ey, ey}], Line[scale {- ez, ez}],
     Tex["x", (scale + δ) ex], Tex["y", (scale + δ) ey], Tex["z", (scale + δ) ez]};

   ℓ = 2 rRing;
   axisOfRotation = {Blue, Tube[{-0.9 ez, 0.9 ez}, 0.01]};
   ωVec = {Red, Arrow@Tube[{O, {0, 0, ℓ / 2} }, 0.02],
     Tex["ω⃗", O + {0, 0, ℓ / 2} + {δ, 0, 0}]};
   ring = {{Directive[Gray, Thick], Tube[
       Table[{rRing Cos[ϕ], rRing Sin[ϕ], rRing}, {ϕ, 0, 2 π, 2 π / 64}], 0.02]}};
   stick = {Directive[Brown, Thick], Tube[{{0, -rRing, ℓ Tan[α] / 2},
       {0, rRing, -ℓ Tan[α] / 2}}, 0.01]};
   markers = {
     (* rotation arrow *) {Arrowheads[Medium],
      Arrow@Table[{0.3 Cos[ϕ], 0.3 Sin[ϕ], 5 ℓ / 8}, {ϕ, π / 4, 2 π, 2 π / 64}]},
     (* αArc *) { Line@Table[StoC[ℓ/4, θ, π / 2], {θ, 0, -α, - α/64}],
      {Black, Text[Stl["α"], StoC[1.2 ℓ/4, -α / 2, π / 2]]}}}};

   LVec =

    G[1] = Graphics3D[{axes, axisOfRotation, ωVec, ring, stick, markers},
      Boxed → False, ImageSize → 320, ViewPoint → vp, PlotRange → range]];

GenerateFigure2a[]
```
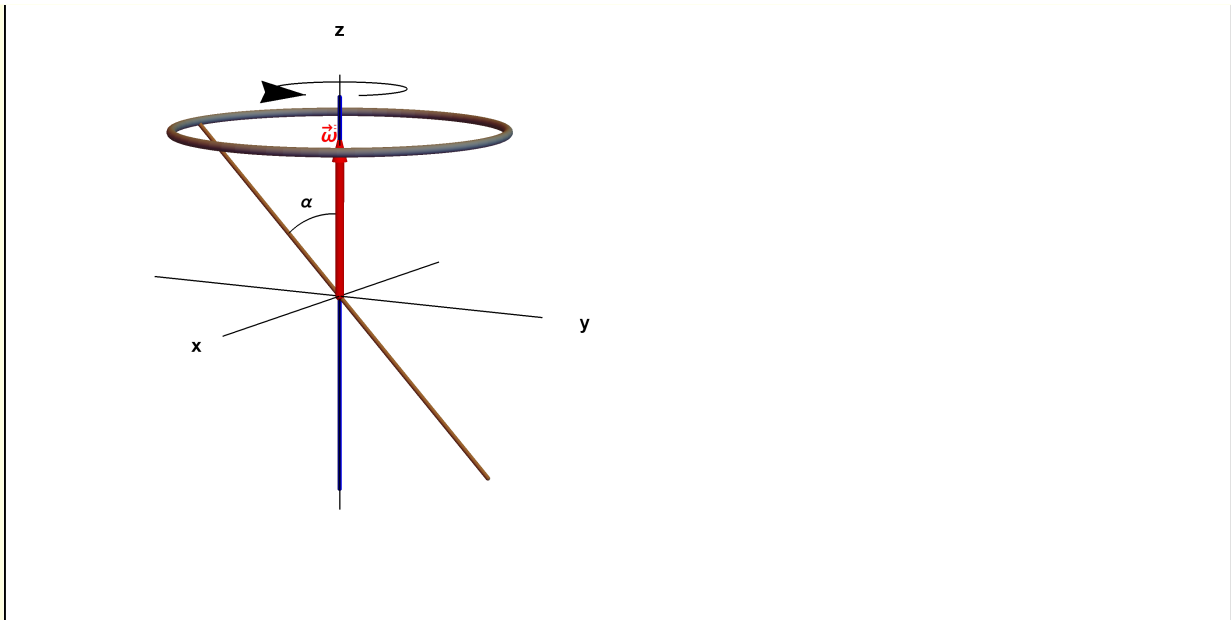
```mathematica
Clear[GenerateFigure2b];
GenerateFigure2b[] :=
  Module[{α = π / 4, rRing = 0.5,  scale = 1.1, δ = 0.1, vp = {4.5, 2.5, 1},
    ℓ, O, ex, ey, ez, e1, e2, e3, axes, axes123, range, P, V, axisOfRotation ,
    ωVec, ring, stick, markers, RM = RotationMatrix, G},
   range = {{- 1, 1}, {- 1, 1}, {- 1,  1}};
   {O, ex, ey, ez} = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
   axes =
    {Black, Line[scale {-ex, ex}], Line[scale { -ey, ey}], Line[scale {- ez, ez}],
     Tex["x", (scale + δ) ex], Tex["y", (scale + δ) ey], Tex["z", (scale + δ) ez]};
   {e1, e2, e3} = {ex, RM[α, e1].ey, RM[α, e1].ez};
   axes123 = {Gray, Arrowheads[Small], Arrow[{-e1, e1}], Tex["1", 1.1 e1],
     Arrow[{-e2, e2}], Tex["2", 1.1 e2], Arrow[{-e3, e3}], Tex["3", 1.1 e3]};

   ℓ = 2 rRing;
   axisOfRotation = {Blue, Tube[{-0.9 ez, 0.9 ez}, 0.01]};
   ωVec = {Red, Arrow@Tube[{O, {0, 0, ℓ / 2} }, 0.02],
     Tex["ω⃗", O + {0, 0, ℓ / 2} + {δ, 0, 0}]};
   ring = {{Directive[Gray, Thick], Tube[
        Table[{rRing Cos[ϕ], rRing Sin[ϕ], rRing}, {ϕ, 0, 2 π, 2 π / 64}], 0.02]}};
   stick = {Directive[Brown, Thick], Tube[{{0, -rRing, ℓ Tan[α] / 2},
       {0, rRing, -ℓ Tan[α] / 2}}, 0.01]};

   markers = {
     (* rotation arrow *) {Arrowheads[Medium],
      Arrow@Table[{0.3 Cos[ϕ], 0.3 Sin[ϕ], 5 ℓ / 8}, {ϕ, π / 4, 2 π, 2 π / 64}]},
     (* αArc *){ Line@Table[StoC[ℓ/4, θ, π / 2], {θ, 0, -α, -α/64}],
      {Black, Text[Stl["α"], StoC[1.2 ℓ/4, -α / 2, π / 2]]}}}};

   G[1] = Graphics3D[{axes, axes123, axisOfRotation, ωVec, ring, stick, markers},
     Boxed → False, ImageSize → 320, ViewPoint → vp, PlotRange → range]];

GenerateFigure2b[]
```
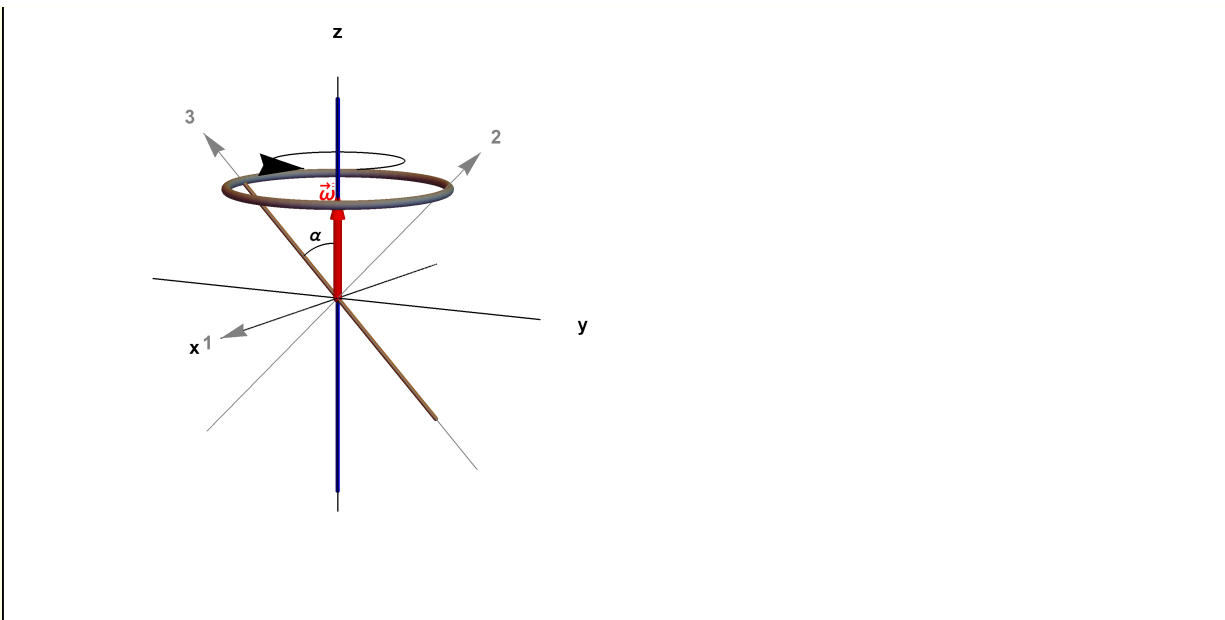
```mathematica
Clear[GenerateFigure2c];
GenerateFigure2c[] :=
  Module[{α = π / 4, rRing = 0.5, LMag = 0.5,  scale = 1.1, δ = 0.1, vp = {4.5, 2.5, 1},
    ℓ, O, ex, ey, ez, e1, e2, e3, axes, axes123, range, P, V, axisOfRotation ,
    ωVec, ring, stick, markers, LVec, LRing, RM = RotationMatrix, G},
   range = {{- 1, 1}, {- 1, 1}, {- 1,  1}};
   {O, ex, ey, ez} = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
   axes =
    {Black, Line[scale {-ex, ex}], Line[scale { -ey, ey}], Line[scale {- ez, ez}],
     Tex["x", (scale + δ) ex], Tex["y", (scale + δ) ey], Tex["z", (scale + δ) ez]};
   {e1, e2, e3} = {ex, RM[α, e1].ey, RM[α, e1].ez};
   axes123 = {Gray, Arrowheads[Small], Arrow[{-e1, e1}], Tex["1", 1.1 e1],
     Arrow[{-e2, e2}], Tex["2", 1.1 e2], Arrow[{-e3, e3}], Tex["3", 1.1 e3]};

   ℓ = 2 rRing;
   axisOfRotation = {Blue, Tube[{-0.9 ez, 0.9 ez}, 0.01]};
   ωVec = {Red, Arrow@Tube[{O, {0, 0, ℓ / 2} }, 0.02],
     Tex["ω⃗", O + {0, 0, ℓ / 2} + {δ, 0, 0}]};
   ring = {{Directive[Gray, Thick], Tube[
       Table[{rRing Cos[ϕ], rRing Sin[ϕ], rRing}, {ϕ, 0, 2 π,  2 π / 64}], 0.02]}};
   stick = {Directive[Brown, Thick], Tube[{{0, -rRing, ℓ Tan[α] / 2},
      {0, rRing, -ℓ Tan[α] / 2}}, 0.01]};

   markers = {
     (* rotation arrow *)(* {Arrowheads[Medium],
      Arrow@Table[{0.3Cos[ϕ],0.3 Sin[ϕ],5ℓ /8}, {ϕ,π/4,2 π, 2 π/64}]},*)
     (* αArc *){ Line@Table[StoC[ℓ/4, θ, π / 2], {θ, 0, -α, -α/64}],
      {Black, Text[Stl["α"], StoC[1.2 ℓ/4, -α/2, π/2]]}}};

   LVec = {Blue, Arrow@Tube[{O, e2 LMag}, 0.02], Tex["L⃗", O + e2 LMag + {δ, 0, 0}]};
   LRing = {Lighter[Blue, 0.5], Arrowheads[Small],
     Arrow@Table[RM[ϕ, ez].(e2 LMag), {ϕ, π, 2 π + π, π/32.}]};


   G[1] = Graphics3D[
     {axes, axes123, axisOfRotation, ωVec, ring, stick, markers, LVec, LRing},
     Boxed → False, ImageSize → 320, ViewPoint → vp, PlotRange → range]];

GenerateFigure2c[]
```

```mathematica
Clear[GenerateFigure2d];
GenerateFigure2d[] :=
  Module[{α = π / 4, rRing = 0.5, LMag = 0.5, dLdtMag = 0.5, τMag = 0.5,
    scale = 1.1, δ = 0.1, vp = {4.5, 2.5, 1}, ℓ, O, ex, ey, ez, e1, e2, e3, axes,
    axes123, range, P, V, axisOfRotation , ωVec, ring, stick, markers, LVec,
    LRing, gravityVec, tensionVec, dLdtVec, τVec, RM = RotationMatrix, G},
   range = {{-1, 1}, {-1, 1}, {-1, 1}};
   {O, ex, ey, ez} = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
   axes =
    {Black, Line[scale {-ex, ex}], Line[scale {-ey, ey}], Line[scale {-ez, ez}],
     Tex["x", (scale + δ) ex], Tex["y", (scale + δ) ey], Tex["z", (scale + δ) ez]};
   {e1, e2, e3} = {ex, RM[α, e1].ey, RM[α, e1].ez};
   axes123 = {Gray, Arrowheads[Small], Arrow[{-e1, e1}], Tex["1", 1.1 e1],
     Arrow[{-e2, e2}], Tex["2", 1.1 e2], Arrow[{-e3, e3}], Tex["3", 1.1 e3]};

   ℓ = 2 rRing;
   axisOfRotation = {Blue, Tube[{-0.9 ez, 0.9 ez}, 0.01]};
   ωVec = {Red, Arrow@Tube[{O, {0, 0, ℓ/2} }, 0.02],
     Tex["ω⃗", O + {0, 0, ℓ/2} + {δ, 0, 0}]};
   ring = {{Directive[Gray, Thick], Tube[
       Table[{rRing Cos[φ], rRing Sin[φ], rRing}, {φ, 0, 2 π, 2 π / 64}], 0.02]}};
   stick = {Directive[Brown, Thick], Tube[{{0, -rRing, ℓ Tan[α] /2},
      {0, rRing, -ℓ Tan[α] /2}}, 0.01]};

   markers = {
     (* rotation arrow *)(* {Arrowheads[Medium],
      Arrow@Table[{0.3Cos[φ],0.3 Sin[φ],5ℓ /8}, {φ,π/4,2 π, 2 π/64}]},*)
     (* αArc *){ Line@Table[StoC[ ℓ/4, θ, π/2], {θ, 0, -α, -α/64}],
      {Black, Text[Stl["α"], StoC[1.2 ℓ/4, -α/2, π/2]]}}}};
```

```
LVec = {Blue, Arrow@Tube[{O, e2 LMag}, 0.02], Tex["L⃗", O + e2 LMag + {δ, 0, 0}]};
LRing = {Lighter[Blue, 0.5], Arrowheads[Small],
   Arrow@Table[RM[ϕ, ez].(e2 LMag), {ϕ, π, 2 π + π, π/32.}]};
gravityVec = With[{st = O, fn = O + {0, 0, -3 δ}},
   {Gray, Vec[{st, fn}, 0.01, 0.03], Tex["M g⃗", fn + {0, 0, -δ}]}];
tensionVec =
 With[{st = {0, -rRing, ℓ Tan[α] /2}, fn = {0, -rRing, ℓ Tan[α] /2} + {0, 0, 3 δ}},
   {Gray, Vec[{st, fn}, 0.01, 0.03], Tex["T", fn + {0, 0, δ}]}];
dLdtVec =
 With[{st = LMag e2, fn = LMag e2 + {-dLdtMag, 0, 0}},

   {Blue, Vec[{st, fn}, 0.01, 0.03], {Black, Tex["dL⃗/dt", st + 1.2 (fn - st)]}}];

τVec =
 With[
   {st = {0, -rRing, ℓ Tan[α] /2}, fn = {0, -rRing, ℓ Tan[α] /2} + {-τMag, 0, 0}},
   {Green, Vec[{st, fn}, 0.01, 0.03], {Black, Tex["τ", st + 1.2 (fn - st)]}}];


G[1] = Graphics3D[{axes, axes123, axisOfRotation, ωVec, ring, stick, markers,
    LVec, LRing, gravityVec, tensionVec, dLdtVec, τVec}, Boxed → False,
   ImageSize → 320, ViewPoint → vp, ViewVertical → {0, 0, 1}, PlotRange → range]];
```

GenerateFigure2d[]

Section 3

```
Clear[GenerateFigure3a];
GenerateFigure3a[] := Module[{scale = 1.1, δ = 0.1, α = π / 4, ℓ = 1,
    mgmag = 0.5, vp = {2.5, 1.0, 1}, O, ex, ey, ez, axesxyz, θP, ϕP, θL, range,
    P, stringAndBob, rotationPath, mgVec, TVec, FcenVec, surface, lab, G},

   range = {{-ℓ, ℓ}, {-ℓ, ℓ}, {-ℓ, ℓ}};
   {O, ex, ey, ez} = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
   axesxyz =
    {Black, Line[scale {-ex, ex}], Line[scale {-ey, ey}], Line[scale {-ez, ez}],
     Tex["x", (scale + δ) ex], Tex["y", (scale + δ) ey], Tex["z", (scale + δ) ez] };

   {θP, ϕP} = {π - α, π / 2};
   (* spherical coordinates for P consistent with origin *)
   P = {0, ℓ Sin[α], -ℓ Cos[α]};
   surface = {Opacity[0.3, Blue],
     Polygon[1 / 2 {{-ℓ, -ℓ, 0}, {-ℓ, ℓ, 0}, {ℓ, ℓ, 0}, {ℓ, -ℓ, 0}}]};

   stringAndBob = {
     { Line[{O, P}], PointSize[0.02], Point[P]}, {Black, Tex["ℓ", P / 2 + {0, δ, δ}]},
     (* vaxis *) {Line[1.1 {-ez, {0, 0, 0}}]},
     (* αArc *) {Line@Table[StoC[ℓ/3, θ, ϕP], {θ, θP, π, θP/64}],
       {Black, Text["α", StoC[1.2 ℓ/3, (π + θP)/2, ϕP]]}}}};
   rotationPath = {{Directive[Gray, Dashed],
       Line[Table[ℓ {Cos[α] Cos[ϕ], Cos[α] Sin[ϕ], -Cos[α]}, {ϕ, 0, 2 π, 2 π / 64}]]},
      {Arrowheads[Small], Arrow[{ℓ {Cos[α] Cos[0], Cos[α] Sin[0], -Cos[α]},
         ℓ {Cos[α] Cos[2 π / 64], Cos[α] Sin[2 π / 64], -Cos[α]}}]}};
   (* vectors *)
   mgVec = With[{st = P, fn = P + {0, 0, -mgmag}, sz = 0.01, szArr = 0.03},
     {Gray, Vec[{st, fn}, sz, szArr], Black, Tex["m g", st + 1.1 (fn - st)]}];
   FcenVec = With[{st = P, fn = P + {0, mgmag, 0}, sz = 0.01, szArr = 0.03},
     {Gray, Vec[{st, fn}, sz, szArr], Black, Tex["Fcen", st + 1.2 (fn - st)]}];
   TVec = With[{st = P, fn = P + {0, -mgmag Cos[α], mgmag Sin[α]}, sz = 0.01, szArr =
        0.03}, {Gray, Vec[{st, fn}, sz, szArr], Black, Tex["T", st + 1.1 (fn - st)]}];

   lab = Stl@StringForm["Spherical pendulum\nF⃗ = m a⃗ method"];
   G = Graphics3D[
     {{axesxyz, rotationPath, stringAndBob, surface, mgVec, TVec, FcenVec}},
     Boxed → False, ImageSize → 400, ViewPoint → vp,
     SphericalRegion → True, PlotLabel → lab]];

GenerateFigure31[]
```
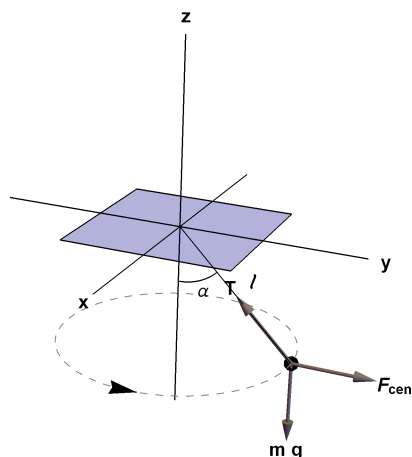
Spherical pendulum
$$\vec{F} = m\,\vec{a}\ \text{method}$$


```
Clear[GenerateFigure3b];
GenerateFigure3b[] :=
  Module[{scale = 1.1, δ = 0.1, α = π / 4, ℓ = 1, mgmag = 0.5, ωmag = 0.5,
    Lmag = 0.5, dLdtmag = 0.5, τmag = 0.5, vp = {2.5, 1.0, 1}, O, ex, ey, ez,
    axesxyz, P, ΘP, ϕP, range, stringAndBob, rotationPath,
    mgVec, LVec, dLdtVec, τVec, ωVec, surface, mess, lab, G},

    range = {{- ℓ, ℓ}, {- ℓ, ℓ}, {- ℓ, ℓ}};
    {O, ex, ey, ez} = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
    axesxyz =
      {Black, Line[scale {-ex, ex}], Line[scale { -ey, ey}], Line[scale {- ez, ez}],
       Tex["x", (scale + δ) ex], Tex["y", (scale + δ) ey], Tex["z", (scale + δ) ez] };

    {ΘP, ϕP} = {π - α, π / 2};
    (* spherical coordinates for P consistent with origin *)
    P = {0, ℓ Sin[α], - ℓ Cos[α]};
    surface = {Opacity[0.3, Blue],
      Polygon[1 / 2 {{- ℓ, - ℓ, 0}, {- ℓ, ℓ, 0}, {ℓ, ℓ, 0}, {ℓ, - ℓ, 0}}]};

    stringAndBob = {
      { Line[{O, P}], PointSize[0.02], Point[P]}, {Black, Tex["ℓ", P / 2 + {0, δ, δ}]},
      (* vaxis *) {Line[1.1 {-ez, {0, 0, 0}}]},
```

```mathematica
   (* αArc *) {Line@Table[StoC[ℓ/3, θ, ϕP], {θ, θP, π, θP/64}],

      {Black, Text["α", StoC[1.2 ℓ/3, (π + θP)/2, ϕP]]}}}};
rotationPath = {{Directive[Gray, Dashed],
   Line[Table[ℓ {Cos[α] Cos[ϕ], Cos[α] Sin[ϕ], -Cos[α]}, {ϕ, 0, 2 π, 2 π / 64}]]},
   {Arrowheads[Small], Arrow[{ℓ {Cos[α] Cos[0], Cos[α] Sin[0], -Cos[α]},
      ℓ {Cos[α] Cos[2 π / 64], Cos[α] Sin[2 π / 64], -Cos[α]}}]}};
(* vectors *)
mgVec = With[{st = P, fn = P + {0, 0, -mgmag}, sz = 0.01, szArr = 0.03},
   {Gray, Vec[{st, fn}, sz, szArr], Black, Tex["m g", st + 1.1 (fn - st)]}];
ωVec = With[{st = O, fn = O + {0, 0, ωmag}, sz = 0.01, szArr = 0.03},
   {Red, Vec[{st, fn}, sz, szArr], Black, Tex["ω", st + 1.2 (fn - st)]}];
LVec = With[{st = O, fn = O + {0, Lmag Cos[α] Sin[α], Lmag Sin[α]^2},
    sz = 0.01, szArr = 0.03},
   {Blue, Vec[{st, fn}, sz, szArr], Black, Tex["L", st + 1.1 (fn - st)]}];
dLdtVec = With[{st = {0, Lmag Cos[α] Sin[α], Lmag Sin[α]^2},
    fn = {0, Lmag Cos[α] Sin[α], Lmag Sin[α]^2} + {-dLdtmag, 0, 0},
    sz = 0.01, szArr = 0.03},
   {Blue, Vec[{st, fn}, sz, szArr], Black, Tex["dLdt", st + 1.2 (fn - st)]}];
τVec = With[{st = P, fn = P + {-τmag, 0, 0}, sz = 0.01, szArr = 0.03},
   {Green, Vec[{st, fn}, sz, szArr], Black, Tex["τ", st + 1.1 (fn - st)]}];
mess = Mess["ref point", O, {0, -5 δ, 2 δ}];

lab = Stl@StringForm["Spherical pendulum\ndL⃗/dt = τ⃗ about pivot point method"];
G = Graphics3D[{{axesxyz, rotationPath, stringAndBob, surface, mgVec,
     LVec, dLdtVec, ωVec, τVec, mess}}, Boxed → False, ImageSize → 400,
   ViewPoint → vp, SphericalRegion → True, PlotLabel → lab]];

GenerateFigure3b[]
```
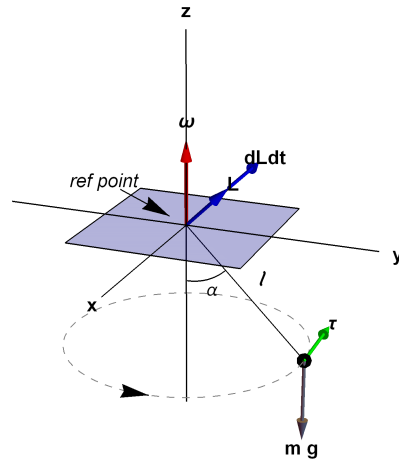
Spherical pendulum
$d\vec{L}/dt = \vec{\tau}$ **about pivot point method**


```
Clear[GenerateFigure3c];
GenerateFigure3c[] :=
  Module[{scale = 1.1, δ = 0.1, α = π / 4, ℓ = 1, mgmag = 0.5, ωmag = 0.5,
    Lmag = 0.5, dLdtmag = 0.5, τmag = 0.5, vp = {2.5, 1.0, 1}, O, ex, ey, ez,
    axesxyz, P, ΘP, ϕP, range, stringAndBob, rotationPath, mgVec, TperpVec,
    TpelVec, FcenVec, LVec, τPerVec, τPelVec, ωVec, surface, mess, lab, G},

    range = {{- ℓ, ℓ}, {- ℓ, ℓ}, {- ℓ, ℓ}};
    {O, ex, ey, ez} = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
    axesxyz =
     {Black, Line[scale {-ex, ex}], Line[scale { -ey, ey}], Line[scale {- ez, ez}],
      Tex["x", (scale + δ) ex], Tex["y", (scale + δ) ey], Tex["z", (scale + δ) ez] };

    {ΘP, ϕP} = {π - α, π / 2};
    (* spherical coordinates for P consistent with origin *)
    P = {0, ℓ Sin[α], - ℓ Cos[α]};
    surface = {Opacity[0.3, Blue],
      Polygon[1 / 2 {{- ℓ, - ℓ, 0}, {- ℓ, ℓ, 0}, {ℓ, ℓ, 0}, {ℓ, - ℓ, 0}}]};

    stringAndBob = {
      { Line[{O, P}], PointSize[0.02], Point[P]}, {Black, Tex["ℓ", P / 2 + {0, δ, δ}]},
      (* vaxis *) {Line[1.1 {-ez, {0, 0, 0}}]},
```
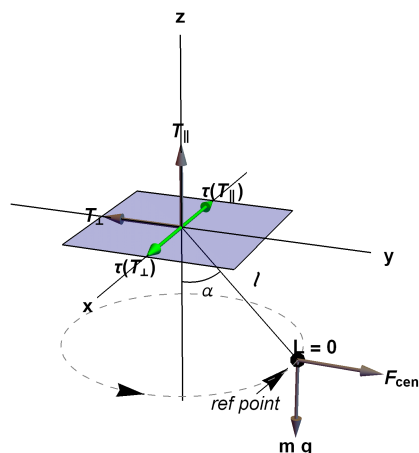
```
      (* αArc *) {Line@Table[StoC[ ℓ/3, θ, φP], {θ, θP, π, θP/64}],
        {Black, Text["α", StoC[1.2 ℓ/3, (π+θP)/2, φP]]}}};
    rotationPath = {{Directive[Gray, Dashed],
      Line[Table[ℓ {Cos[α] Cos[φ], Cos[α] Sin[φ], -Cos[α]}, {φ, 0, 2 π, 2 π / 64}]]},
     {Arrowheads[Small], Arrow[{ℓ {Cos[α] Cos[0], Cos[α] Sin[0], -Cos[α]},
        ℓ {Cos[α] Cos[2 π / 64], Cos[α] Sin[2 π / 64], -Cos[α]}}]}};
    (* vectors *)
    mgVec = With[{st = P, fn = P +{0, 0, -mgmag}, sz = 0.01, szArr = 0.03},
      {Gray, Vec[{st, fn}, sz, szArr], Black, Tex["m g", st + 1.1 (fn - st)]}];
    TpelVec = With[{st = O, fn = O + {0, 0, mgmag}, sz = 0.01, szArr = 0.03},
      {Gray, Vec[{st, fn}, sz, szArr], Black, Tex["T∥", st + 1.1 (fn - st)]}];
    TperpVec = With[{st = O, fn = O + {0, -mgmag, 0}, sz = 0.01, szArr = 0.03},
      {Gray, Vec[{st, fn}, sz, szArr], Black, Tex["T⊥", st + 1.1 (fn - st)]}];
    FcenVec = With[{st = P, fn = P + {0, mgmag, 0}, sz = 0.01, szArr = 0.03},
      {Gray, Vec[{st, fn}, sz, szArr], Black, Tex["Fcen", st + 1.2 (fn - st)]}];
    ωVec = With[{st = O, fn = O + {0, 0, ωmag}, sz = 0.01, szArr = 0.03},
      {Red, Vec[{st, fn}, sz, szArr], Black, Tex["ω", st + 1.2 (fn - st)]}];
    LVec = With[{st = P, fn = P + {-τmag , 0, 0}, sz = 0.01, szArr = 0.03},
      { Black, Tex["L = 0", st + {0, δ, δ}]}];
    τPelVec = With[{st = O, fn = 0 + {-τmag , 0, 0}, sz = 0.01, szArr = 0.03},
      {Green, Vec[{st, fn}, sz, szArr], Black, Tex["τ(T∥)", st + 1.3 (fn - st)]}];
    τPerVec = With[{st = O, fn = 0 + {τmag, 0, 0}, sz = 0.01, szArr = 0.03},
      {Green, Vec[{st, fn}, sz, szArr], Black, Tex["τ(T⊥)", st + 1.3 (fn - st)]}];
    mess = Mess["ref point", P, {6 δ, -δ, 0}];

    lab = Stl@StringForm["Spherical pendulum\ndL⃗/dt = τ⃗ about mass method" ];
    G = Graphics3D[{{axesxyz, rotationPath, stringAndBob, surface, mgVec, FcenVec,
        TpelVec, TperpVec, LVec, τPelVec, τPerVec, mess}}, Boxed → False,
      ImageSize → 400, ViewPoint → vp, SphericalRegion → True, PlotLabel → lab]];

GenerateFigure3c[]
```

Spherical pendulum

$$d\vec{L}/dt = \vec{\tau} \text{ about mass method}$$

Section 4

```
Clear[GenerateFigure4a];
GenerateFigure4a[] :=
  Module[{scale = 1.1, δ = 0.1, α = π / 6, ℓ = 1, h = 0.75, r = 0.1, mgmag = 0.5,
    vp = {2.5, 1.0, 1}, O, ex, ey, ez, axesxyz, range, PCone, PL, RCOM,
    lab, gCone, gRing, gPoints, ωConeVec, ωRingVec, G},

   range = ℓ {{-1, 1}, {-1, 1}, {-0.25, 1}};
   {O, ex, ey, ez} = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
   axesxyz =
    {Black, Line[scale {-ex, ex}], Line[scale {-ey, ey}], Line[scale {-0.25 ez, ez}],
     Tex["x", (scale + δ) ex], Tex["y", (scale + δ) ey], Tex["z", (scale + δ) ez] };

   PCone = {0, 0, ℓ};
   gCone = {{Opacity[0.1, Green], Cone[{PCone, O}, ℓ Tan[α]]},

      (* αArc *) {Black, Line@Table[StoC[ℓ/3, θ, π / 2], {θ, 0, α, α/64}],

       {Black, Text["α", StoC[1.2 ℓ/3, α / 2, π / 2]]}},

      (*hLevel*) {Directive[Black, Dashed],
       Line@Table[h {Tan[α] Sin[φ], Tan[α] Cos[φ], 1}, {φ, 0, 2 π, 2 π / 64}]},
```

```
      (* rotation direction *)
      {Arrowheads[Small], Arrow[{{h Tan[α] Cos[0], h Tan[α] Sin[0], h},
         {h Tan[α] Cos[2 π / 64], h Tan[α] Sin[2 π / 64], h}}]},
      (* ωConeVec *)
     With[{st = {0, 0, h}, fn = {0, 0, h + 0.15}, sz = 0.01, szArr = 0.02},
       {Red, Vec[{st, fn}, sz, szArr], Black, Tex["ω_cone", st + 1.2 (fn - st)]}],
      (*h marker *)
      {Black,
       With[{st = h {Tan[α] Sin[π], Tan[α] Cos[π], 0},
          fn = h {Tan[α] Sin[π], Tan[α] Cos[π], 1}},
        DimMarker3[{st, fn}, "h", 0.5, {0, -δ, 0}]]}};

   (* generate ring at level h *)
   gRing = {{Orange, Tube@Table[{r Sin[ϕ], r Cos[ϕ], h}, {ϕ, 0, 2 π, 2 π / 64}]},
      (* ωRingVec *)
      With[{st = {0, 0, h}, fn = {0, 0, h - 0.3}, sz = 0.01, szArr = 0.02},
       {Red, Vec[{st, fn}, sz, szArr], Black, Tex["ω_ring", st + 1.2 (fn - st)]}]};
   (* translate to edge of cone *)gRing = GeometricTransformation[
     gRing, TranslationTransform[{0, h Tan[α] - r, 0}]];
   (* rotate till plane of ring is perpendiculare to cone surface *)
   gRing = GeometricTransformation[
     gRing, RotationTransform[-α, {1, 0, 0}, {0, h Tan[α], h}]];
   PL = {0, 0, h};
   RCOM = {0, h Tan[α] - r Cos[α], h + r Sin[α] };
   gPoints = {PointSize[0.005], {Black, Point[PL]}, {Red, Point[RCOM]}};

   lab = Stl@StringForm["Ring rolling in cone"];
   G = Graphics3D[{axesxyz, gCone, gRing, gPoints},
     Boxed → False, ImageSize → 600, ViewPoint → vp,
     SphericalRegion → True, ViewVertical → {0, 0, 1}, PlotLabel → lab]];

GenerateFigure4a[]
```
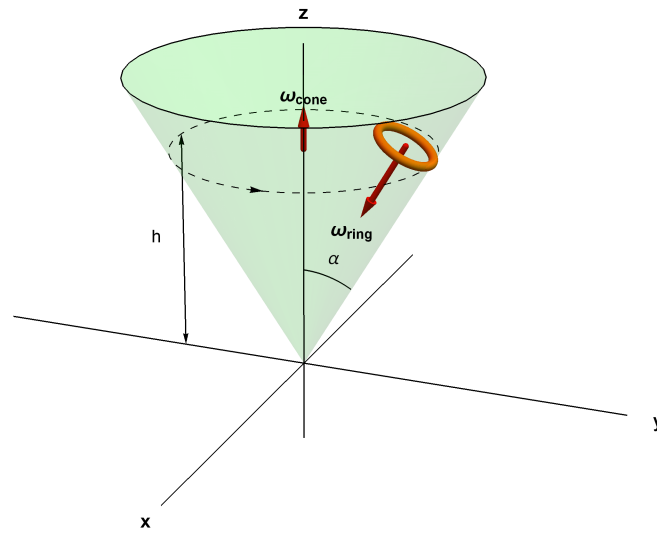
Ring rolling in cone

```
Clear[GenerateFigure4b];
GenerateFigure4b[] := Module[
    {scale = 1.1, O, δ = 0.1, α = π / 6, ℓCone = 1, h = 0.75, a = 0.1, ey, ez, e2, e3,
     axesyz, axes23, yRing, zRing, gCone, gRing, gLines, range, RM = RotationMatrix},

    range = {{- 0.25, 0.5}, {- 0.25, 1}};
    {O, ey, ez} = {{0, 0}, {1, 0}, {0, 1}};
    axesyz = {Black, Line[scale { -0.25 ey, 0.5 ey}], Tex["y", 0.5 (scale + δ) ey],
      Line[scale {-0.25 ez, ez}], Tex["z", (scale + δ) ez] };

    (* construct the unit vectors for the 2D principal axes *)
    {e2, e3} = {RM[-α].ey, RM[-α].ez};
    (* for convenience,
    construct the graphical representation of the axes at the origin O of the
     space coordinates and then translate them to the center of the ring *)
    axes23 = {Blue, Line[0.25 scale { -e2, e2}], Line[0.25 scale {-e3, e3}],
      Tex["2", 0.25 (scale + δ) e2], Tex["3", 0.25 (scale + δ) e3] };
    yRing = h Tan[α] - a Cos[α];
    zRing = h + a Sin[α];
    axes23 = GeometricTransformation[axes23, TranslationTransform[{yRing, zRing}]];

    gCone = {{Green, Line[{O, ℓCone {Sin[α], Cos[α]}}]},
      {Directive[Black, Dashed], Line[{{0, h }, {h Tan[α], h }}]},
      (*αArc*)
      With[{θ1 = π / 2 - α, θ2 = π / 2, r = h / 4},
       {Black, Circle[O, r, {θ1, θ2}],

         Text["α", 1.2 {r Cos[ θ1 + θ2 / 2 ], r Sin[ θ1 + θ2 / 2 ]}]}],

      (* hMarker *)
      {Black, DimMarker3[{{-δ, 0}, {-δ, h}}, "h", 0.5, {-0.02, 0}]}};

    (* generate ring at space coordinate point {0, h} *)
    gRing = {Directive[Orange, Thick],
      Line[{{-a, h}, {a, h }}], PointSize[0.02], Point[{0, h}],
      {Black, DimMarker3[{{0, h + 0.02}, {a, h + 0.02}}, "a", 0.5, {0.0, 0.04}]}};
    (* translate to edge of cone *)
    gRing = GeometricTransformation[gRing, TranslationTransform[{h Tan[α] - a, 0}]];
    (* rotate till plane of ring is perpendicular to cone surface *)
    gRing = GeometricTransformation[gRing, RotationTransform[-α, {h Tan[α], h}]];

    gLines = With[{st = {0, zRing}, fn = {yRing, zRing }},
      {Black, DimMarker3[{st, fn}, "d", 0.5, {0.0, 0.03}], Tex["x ⊙", O + {-δ, -δ}],
       {Blue, Tex["1 ⊙", {yRing, zRing} + {-δ 0.5, δ}]}}];

    Graphics[{axesyz, axes23, gCone, gRing, gLines}]];

GenerateFigure4b[]
```
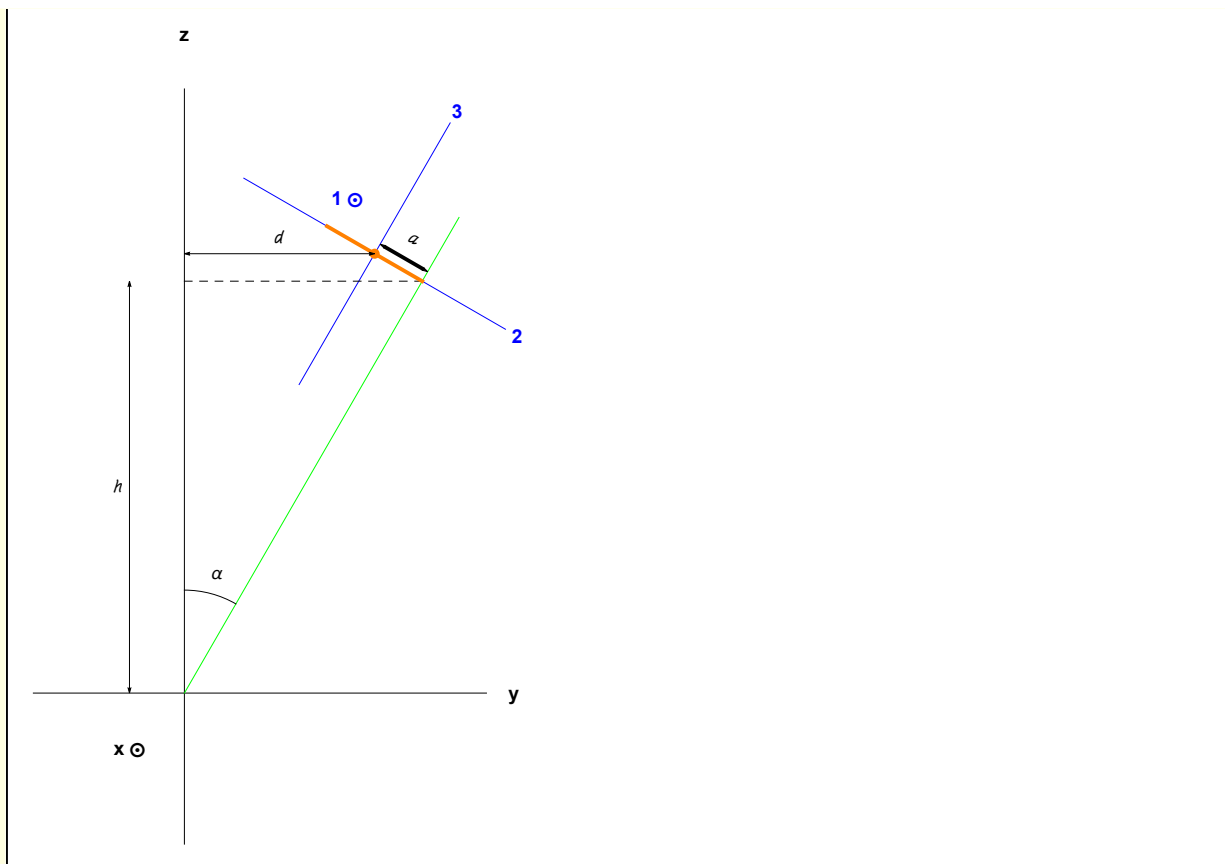
```
Clear[GenerateFigure4c];
GenerateFigure4c[] :=
  Module[{scale = 1.1, δ = 0.1, α = π / 6, ℓ = 1, h = 0.75, r = 0.1, mgmag = 0.5,
    vp = {2.5, 1.0, 1}, O, ex, ey, ez, axesxyz, range, PCone, PL, RCOM,
    lab, gCone, gRing, gPoints, ωConeVec, ωRingVec, G},

   range = ℓ {{- 1, 1}, {- 1, 1}, {- 0.25, 1}};
   {O, ex, ey, ez} = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
   axesxyz =
    {Black, Line[scale {-ex, ex}], Line[scale { -ey, ey}], Line[scale {- 0.25 ez, ez}],
     Tex["x", (scale + δ) ex], Tex["y", (scale + δ) ey], Tex["z", (scale + δ) ez] };

   PCone = {0, 0, ℓ};
   gCone = {{Opacity[0.1, Green], Cone[{PCone, O}, ℓ Tan[α]]},
      (* αArc *) {Black, Line@Table[StoC[ℓ/3, θ, π / 2], {θ, 0, α, α/64}],

       {Black, Text["α", StoC[1.2 ℓ/3, α / 2, π / 2]]}},

      (*hLevel*) {Directive[Black, Dashed],
       Line@Table[h {Tan[α] Sin[φ], Tan[α] Cos[φ], 1}, {φ, 0, 2 π, 2 π / 64}]},
      (* rotation direction *)
      {Arrowheads[Small], Arrow[{{h Tan[α] Cos[0], h Tan[α] Sin[0], h},
         {h Tan[α] Cos[2 π / 64], h Tan[α] Sin[2 π / 64], h}}]},
      (* ωConeVec *)
```

```
      With[{st = {0, 0, h}, fn = {0, 0, h + 0.15}, sz = 0.01, szArr = 0.02},
        {Red, Vec[{st, fn}, sz, szArr], Tex["ωc", st + 1.2 (fn - st)]}]],
      (*h marker *)
      {Black,
       With[{st = h {Tan[α] Sin[π], Tan[α] Cos[π], 0},
          fn = h {Tan[α] Sin[π], Tan[α] Cos[π], 1}},
        DimMarker3[{st, fn}, "h", 0.5, {0, -δ, 0}]]}};

    (* generate ring at level h *)
    gRing = {{Orange, Tube@Table[{r Sin[φ], r Cos[φ], h}, {φ, 0, 2 π, 2 π / 64}]},
      (* L Vec *)
      With[{st = {0, 0, h}, fn = {0, 0, h - 0.2}, sz = 0.01, szArr = 0.02},
        {Blue, Vec[{st, fn}, sz, szArr], Tex["L⃗", st + 1.0 (fn - st) + {0, δ, 0}]}],
      (* dL/dt Vec *)
      With[{st = {0, 0, h - 0.2}, fn = {-0.2, 0, h - 0.2}, sz = 0.01, szArr = 0.02},
        {Blue, Vec[{st, fn}, sz, szArr], Tex["dL⃗/dt", st + 1.4 (fn - st) ]}],
      (* ωRingVec *)
      With[{st = {0, 0, h}, fn = {0, 0, h - 0.3}, sz = 0.01, szArr = 0.02},
        {Red, Vec[{st, fn}, sz, szArr], Tex["ωr", st + 1.2 (fn - st)]}]};
    (* translate to edge of cone *)gRing = GeometricTransformation[
      gRing, TranslationTransform[{0, h Tan[α] - r, 0}]];
    (* rotate till plane of ring is perpendiculare to cone surface *)
    gRing = GeometricTransformation[
      gRing, RotationTransform[-α, {1, 0, 0}, {0, h Tan[α], h}]];
    PL = {0, 0, h};
    RCOM = {0, h Tan[α] - r Cos[α], h + r Sin[α] };
    gPoints = {PointSize[0.005], {Black, Point[PL]}, {Red, Point[RCOM]}};

    lab = Stl@StringForm["Showing ω⃗, L⃗, and dL⃗/dt vectors "];
    G = Graphics3D[{axesxyz, gCone, gRing, gPoints},
      Boxed → False, ImageSize → 600, ViewPoint → vp,
      SphericalRegion → True, ViewVertical → {0, 0, 1}, PlotLabel → lab]];

GenerateFigure4c[]
```
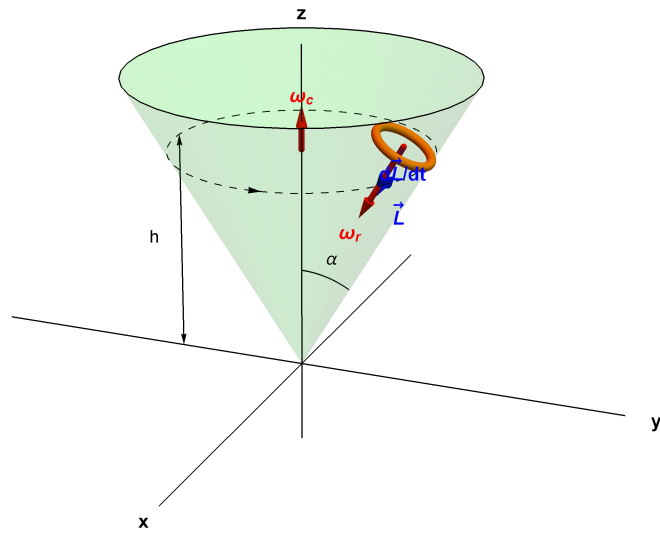
Showing $\vec{\omega}$, $\vec{L}$, and $d\vec{L}/dt$ vectors

```
Clear[GenerateFigure4d];
GenerateFigure4d[] := Module[{scale = 1.1, O, δ = 0.1, αCone = π / 6, ℓCone = 1,
    h = 0.75, rRing = 0.1, P, d, ey, ez, axesyz, gCone, gP, gLines, gVecs, range},

   range = {{- 0.25, 0.5}, {- 0.25, 1}};
   {O, ey, ez} = {{0, 0}, {1, 0}, {0, 1}};
   axesyz = {Black, Line[scale { -0.25 ey, 0.5 ey}], Line[scale {-0.25 ez, ez}],
     Tex["y", 0.5 (scale + δ) ey], Tex["z", (scale + δ) ez] };

   gCone = {{Green, Line[{O, 1.2 ℓCone {Sin[αCone], Cos[αCone]}}]},
     {Directive[Black, Dashed], Line[{{0, h}, {h Tan[αCone], h }}]},
     (*αArc*)
     With[{θ1 = π / 2 - αCone, θ2 = π / 2, r = h / 4},
       {Black, Circle[O, r, {θ1, θ2}],
         Text["α", 1.2 {r Cos[θ1 + θ2 / 2], r Sin[θ1 + θ2 / 2]}]}]],
     (* hMarker *)
     {Black, DimMarker3[{{-δ, 0}, {-δ, h}}, "h", 0.5, {-0.02, 0}]}};

   d = h Tan[αCone];
   P = {d, h};
   gP = {PointSize[0.02], Point[P]};

   gLines = With[{st = {0, h }, fn = {h Tan[αCone], h }},
     {Black, DimMarker3[{st, fn}, "d", 0.5, {0.0, 0.03}]}];


   gVecs = {Directive[Black, Thick],
     With[{st = P, fn = P + {0, -0.1}},
       {Arrow[{st, fn}], Text["m g", st + 1.2 (fn - st)]}],
     With[{st = P, fn = P + {0.1, 0}},
       {Arrow[{st, fn}], Text["F_c", st + 1.2 (fn - st)]}],
     With[{st = P, fn = P + 0.1 h {Sin[αCone], Cos[αCone]}},
       {Arrow[{st, fn}], Text["F_f", st + 1.4 (fn - st)]}],
     With[{st = P, fn = P + 0.1 h {-Cos[αCone], Sin[αCone]}},
       {Arrow[{st, fn}], Text["N", st + 1.4 (fn - st)]}]};

   Graphics[{axesyz, gCone, gP, gLines, gVecs}]];

GenerateFigure4d[]
```
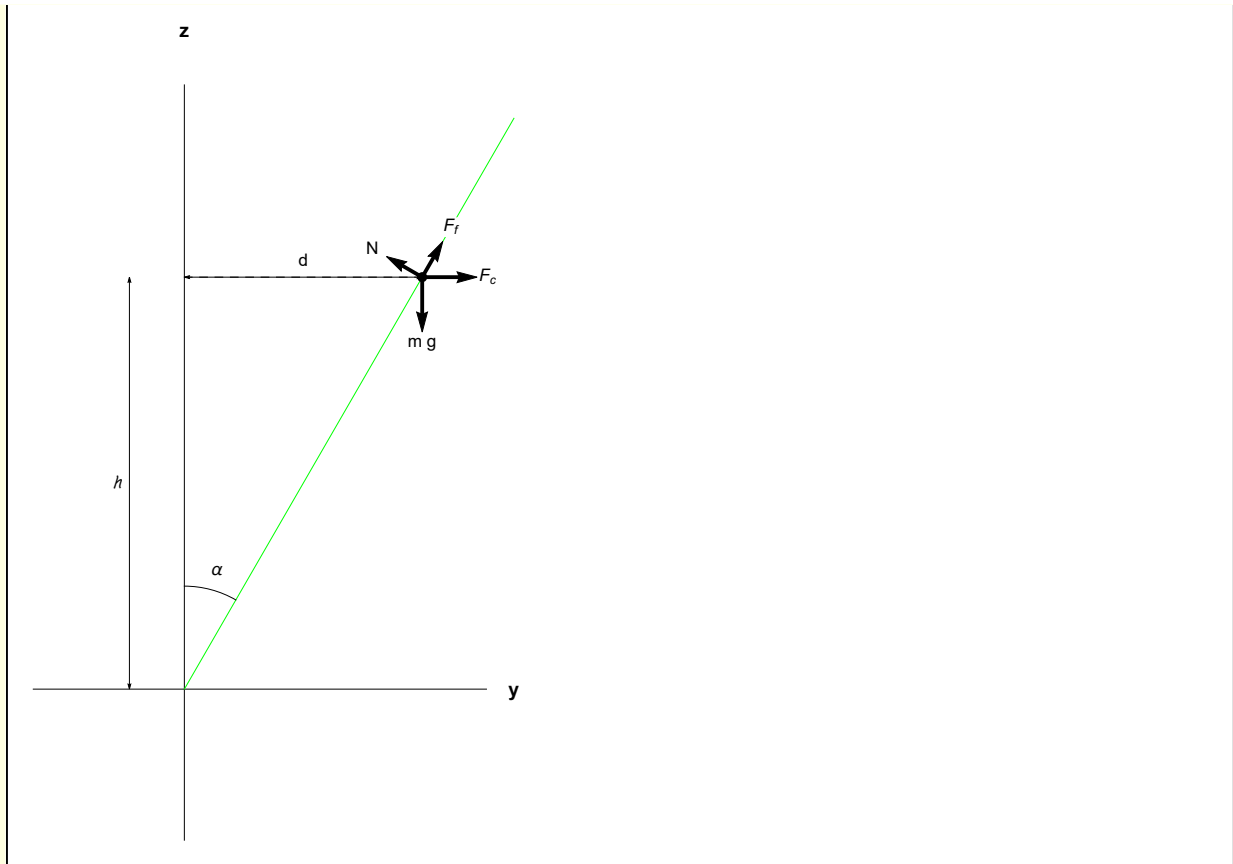
Graphics Utilities

```
(* Helper Functionsc*)
Clear[StoC, Tex, Vec, VecLab, Mess, DimMarker3];
StoC[r_, θ_, ϕ_] := {r Sin[θ] Cos[ϕ], r Sin[θ] Sin[ϕ], r Cos[θ] };
Tex[text_, position_] := Text[Style[text, Bold, FontSize → 10], position];
Vec[vec_] := {Arrowheads[0.05], Arrow[Tube[vec, 0.02]]};
Vec[vec_, size_, sizeAH_] := {Arrowheads[sizeAH], Arrow[Tube[vec, size]]};
(* Draw vector with label place beyond the tip *)
VecLab[st_, fn_, sz_, szArrow_, txt_, txtScale_] :=
 Module[{vec, vecLabel},
  vec = Vec[{st, fn}, sz, szArrow];
  vecLabel = Text[Stl[txt], st + txtScale (fn - st)];
  {vec, vecLabel}]
Mess[lab_, Pobj_, offset_] :=
  Module[{Ptex = Pobj + offset, gText, pointer, dirVec, arrow},
   dirVec = Pobj - Ptex;
   arrow = {Arrowheads[Small], Arrow[{Ptex + 0.2 dirVec, Ptex + 0.8 dirVec}]};
   gText = {Black, Style[Text[lab, Ptex], 10, Italic], arrow}];
DimMarker3[{tail_, head_}, lab_, frac_, offset_] :=
 Module[{labPosn},
  labPosn = tail + frac (head - tail);
  {Arrowheads[{-0.01, 0.01}], Arrow[{tail, head}], Text[lab, labPosn + offset]} ]
```