
Mechanics - Morin Problem 9.3 - Rolling Cone 10-28-18

N. T. Gladd

Initialization: Be sure the file *NTGUtilityFunctions.m* is in the same directory as that from which this notebook was loaded. Then execute the cell immediately below by mousing left on the cell bar to the right of that cell and then typing “shift” + “enter”. Respond “Yes” in response to the query to evaluate initialization cells.

```
SetDirectory[NotebookDirectory[]];  
(* set directory where source files are located *)  
Get["NTGUtilityFunctions.m"]; (* Load utilities package *)
```

Background

This is the second in a series of Mathematica notebooks on classical mechanics. This series was motivated by a close reading and problem solving project I undertook in 2014. The focus of my attention was the text *Introduction to Classical Mechanics with Problems and Solutions*, by David Morin. This is a good book from which to learn and has a great collection of problems. I purchased it and recommend that those with interests in this topic acquire it for their library. I do note that an earlier version can be found on the web. This year, when I returned to this project, I decided to focus on generating Mathematica notebooks on material covered in *Chapter 9 Angular Momentum, Part II (General \vec{L})*, which deals the 3-D rigid body dynamics. This topic is notorious difficult/confusing for students and I felt I just skimmed by as a graduate student. I return in retirement after all these years to pay my dues and really understand how to solve problems in this area.

Although Morin's Chapter 9 guides these notebooks, I made frequent use of other sources such as textbooks available in libraries or on the web. I also found lots of video lectures available on YouTube. Confused about some physics topics? Google it and you'll be amazed what you find. Some relevant texts are

Classical Mechanics, Hebert Goldstein (my original text at University, late 60s). Newer versions exist.

Mechanics: Volume 1 A Course in Theoretical Physics, L. D. Landau and E. M. Lifshitz.

Classical Mechanics, John. R. Taylor

Classical Mechanics of Particles and Systems, Stephen T. Thornton, Jerry B. Marion

Analytical Mechanics, G. R. Fowles, G. L. Cassiday

Analytical Mechanics, Louis N. Hand, Janet D. Finch

I find Mathematica useful for this topic. It facilitates calculations, provides a vehicle for creating instructive visualizations and allows one to quickly generate numerical solutions. Mathematica is a favorite tool of mine but I think it is crucially important to also work with pen and paper. Our brains are closely linked to our hands and one thinks differently with a pen in hand than when sitting before a computer screen. For serious thoughts on this, read *The Craftsman*, by Richard Sennett.

Purpose

This notebook is a take on Morin 9.3, a cone rolling on a table. The problem calls for determination of the angular momentum

Analysis

Consider a cone rotating on a table, as illustrated in the following figure.

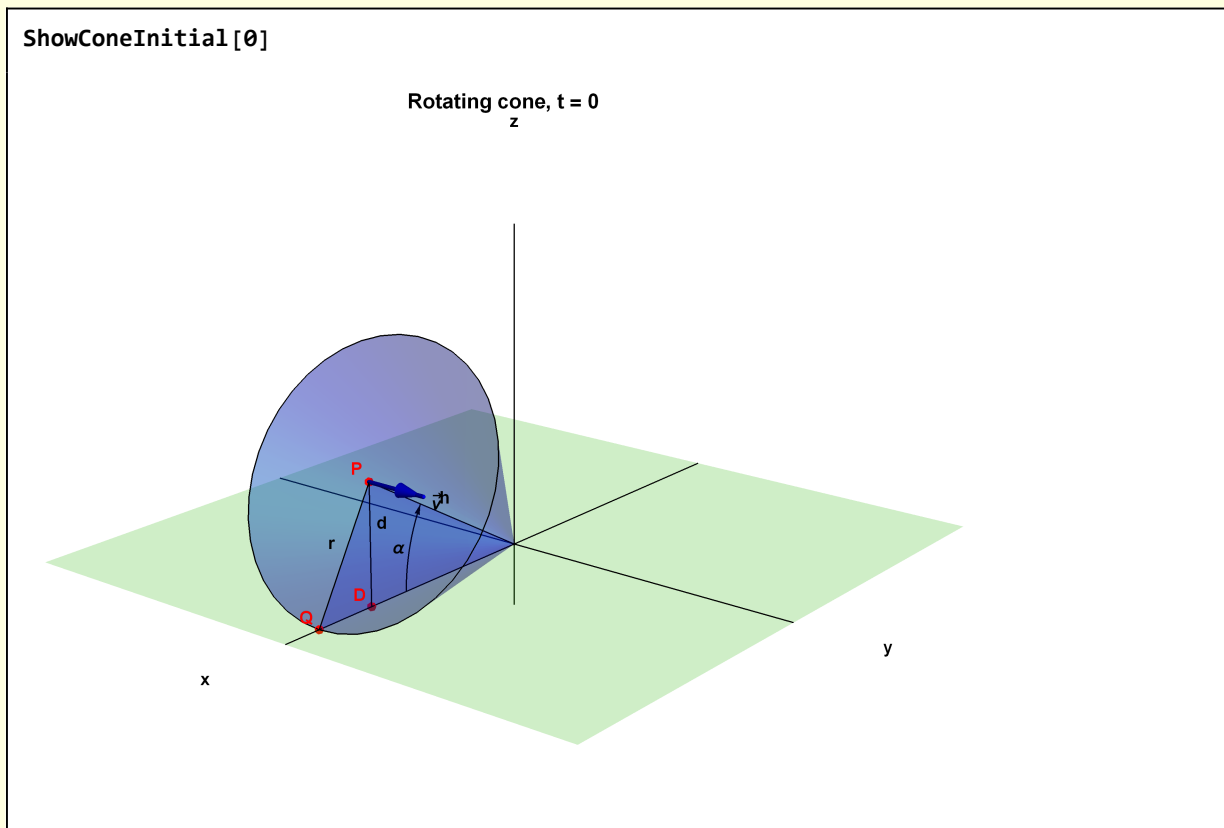
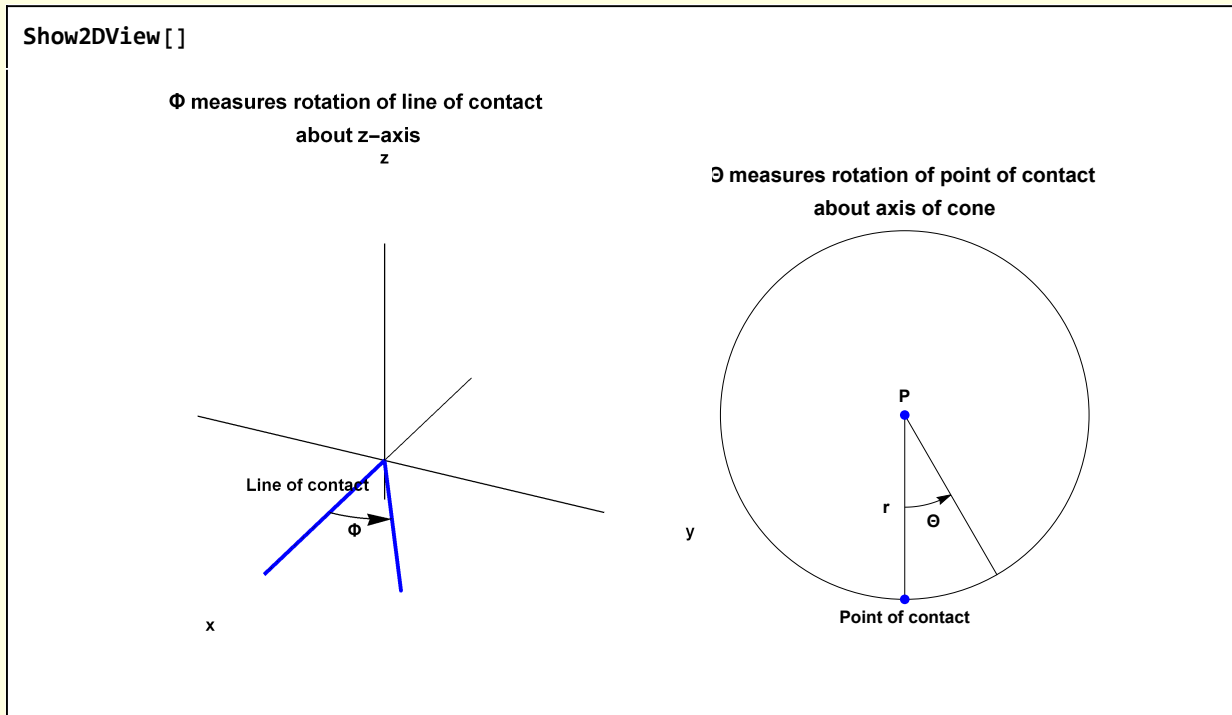


Figure 1 Cone rolling on a plane

Introduce some angular parameters Θ and Φ describing the motion of the cone. I use capital Greek letters to distinguish these parameters from the usual θ and ϕ of spherical coordinates.

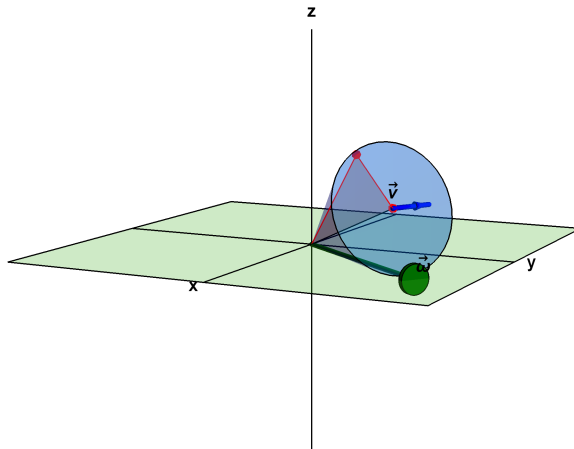


Solution I) Geometrical arguments

Morin comments that $\vec{\omega}$ points along the line of contact of the cone with the surface since the points on this line have no instantaneous motion.

```
With[{t = 1},
ShowConeSolution1[t]]
```

Rotating cone, t = 1.00



To determine the magnitude $|\vec{\omega}|$ note that as P moves a distance $v dt$, the contact point of the cone rolls without slipping through an angle $(h \sin(\alpha)) d\Theta$.

$$v dt = h \sin(\alpha) d\Theta$$

Then

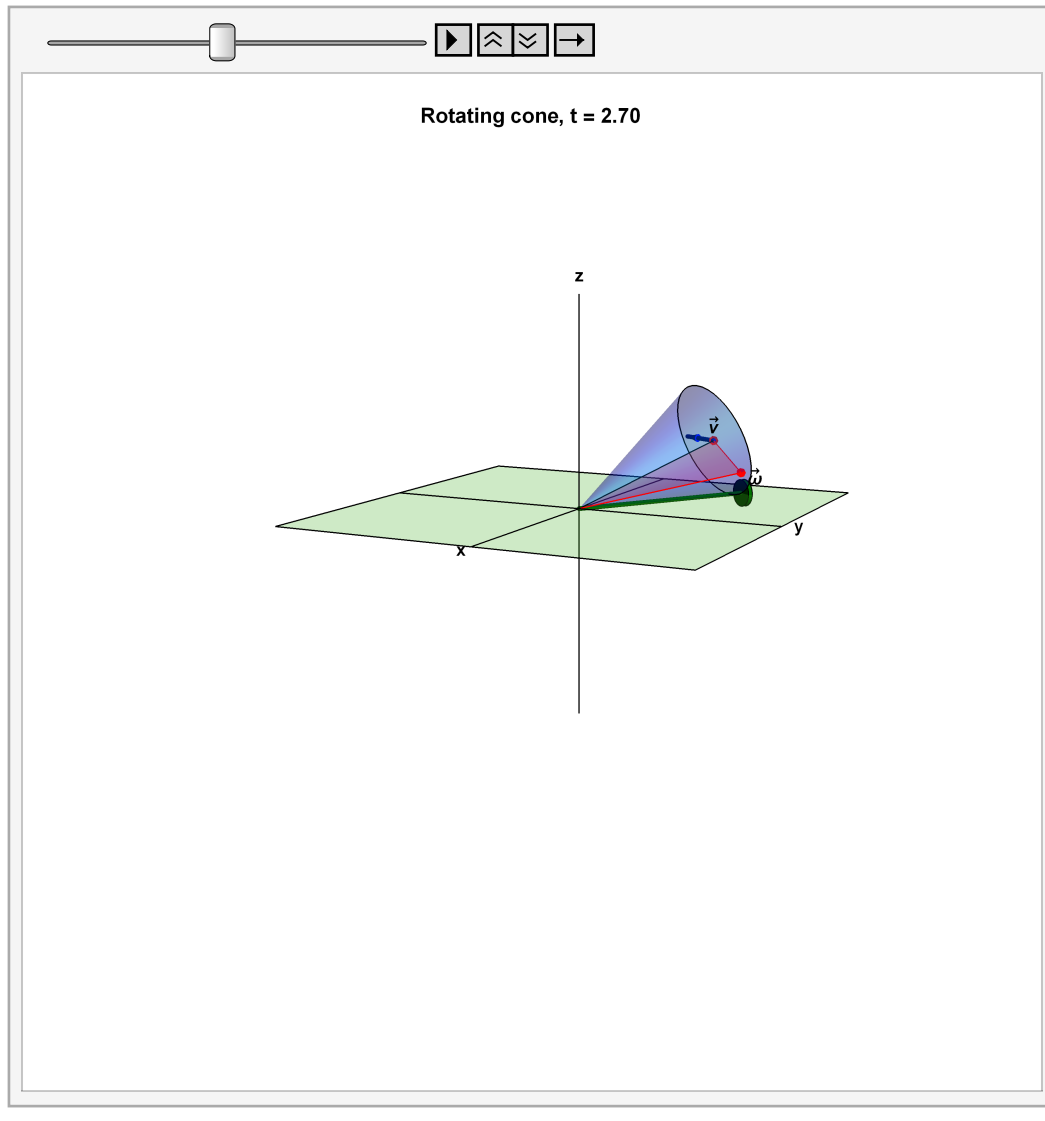
$$\omega = \frac{d\Theta}{dt} = \frac{v}{h \sin(\alpha)}$$

Thus, at $t = 0$

$$\vec{\omega} = \left(\frac{v}{h \sin(\alpha)} \right) \{1, 0, 0\}$$

I animate the solution

```
Module[{tMax = 6, dt = 0.1, frames},
frames = Table[ShowConeSolution1[t], {t, 0, tMax, dt}];
ListAnimate[frames]]
```



Solution using relative frames of reference

In this case, Morin makes use of relative frames of reference S_1 , S_2 , S_3 (see the text). The S_1 frame is fixed in the cone, the S_2 frame is a rotating frame, and S_3 is the lab frame. The relative angular momenta are $(\vec{\omega}_{1,2})$ oriented along the axis of the cone, and $(\vec{\omega}_{2,3})$ which is oriented as shown in the figure. We must have

and

$$\vec{v} = r \hat{1}_r \times \vec{\omega}_{2,3}$$

From addition of the angular velocity vectors

$$\begin{aligned} |\omega_{1,3}| &= \sqrt{|\omega_{1,2}|^2 + |\omega_{2,3}|^2} = \sqrt{\left(\frac{v}{h}\right)^2 + \left(\frac{v}{h \tan(\alpha)}\right)^2} = \left(\frac{v}{h}\right) \sqrt{\frac{1}{\tan(\alpha)^2} + 1} \\ &= \left(\frac{v}{h}\right) \sqrt{\frac{\cos(\alpha)^2}{\sin(\alpha)^2} + 1} = \left(\frac{v}{h}\right) \sqrt{\frac{\cos(\alpha)^2 + \sin(\alpha)^2}{\sin(\alpha)^2}} \\ &= \left(\frac{v}{h}\right) \frac{1}{\sin(\alpha)} \end{aligned}$$

which is the same as solution 1

Visualizations

```

Clear[ShowConeInitial];
ShowConeInitial[t_] :=
Module[{scale = 1.25, δ = 0.05, vp = {2.5, 2.0, 1}, v = 1, h = 0.75, α = π/6, 0,
  ex, ey, ez, gAxes, range, gPlaneBase, θ, φ, ϑ, ϕ, ϑ0, ϕ0, rCone, dCone, gCone,
  ptPCone, ptQCone, ptDCone, gConeMarkers, gConeMarkersRotated, vVec, lab, StoC},
StoC[r_, θ_, φ_] := {r Sin[θ] Cos[φ], r Sin[θ] Sin[φ], r Cos[θ]};
range = {{-1, 1}, {-1, 1}, {-0.2, 1}};
{0, ex, ey, ez} = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
gAxes = {Black, Line[scale {-ex, ex}], Line[scale {-ey, ey}],
  Line[scale {-ez, ez}], Stl@Text["x", (scale + δ) ex],
  Stl@Text["y", (scale + δ) ey], Stl@Text["z", (scale + δ) ez]};
gPlaneBase =
{Opacity[0.25, Lighter[Green, 0.25]],
  Polygon[scale {{-1, -1, 0}, {1, -1, 0}, {1, 1, 0}, {-1, 1, 0}}]};

(* initial angle between z-axis and rotation axis of the cone *)
θ = π/2 - α;
ϑ0 = 0; (* cone is initially oriented along x-axis *)
ϑ0 = 0; (* initial line between center of cone and point of contact *)
ϑ = ϑ0 +  $\frac{v}{h \tan[\alpha]}$  t;
ϕ = ϕ0 + Sin[α] ϑ;
φ = ϕ;
(* half-width of base of cone *)
rCone = h Tan[α];
(* *)
dCone = h Cos[α];
(* point at center of base of cone *)

```

```

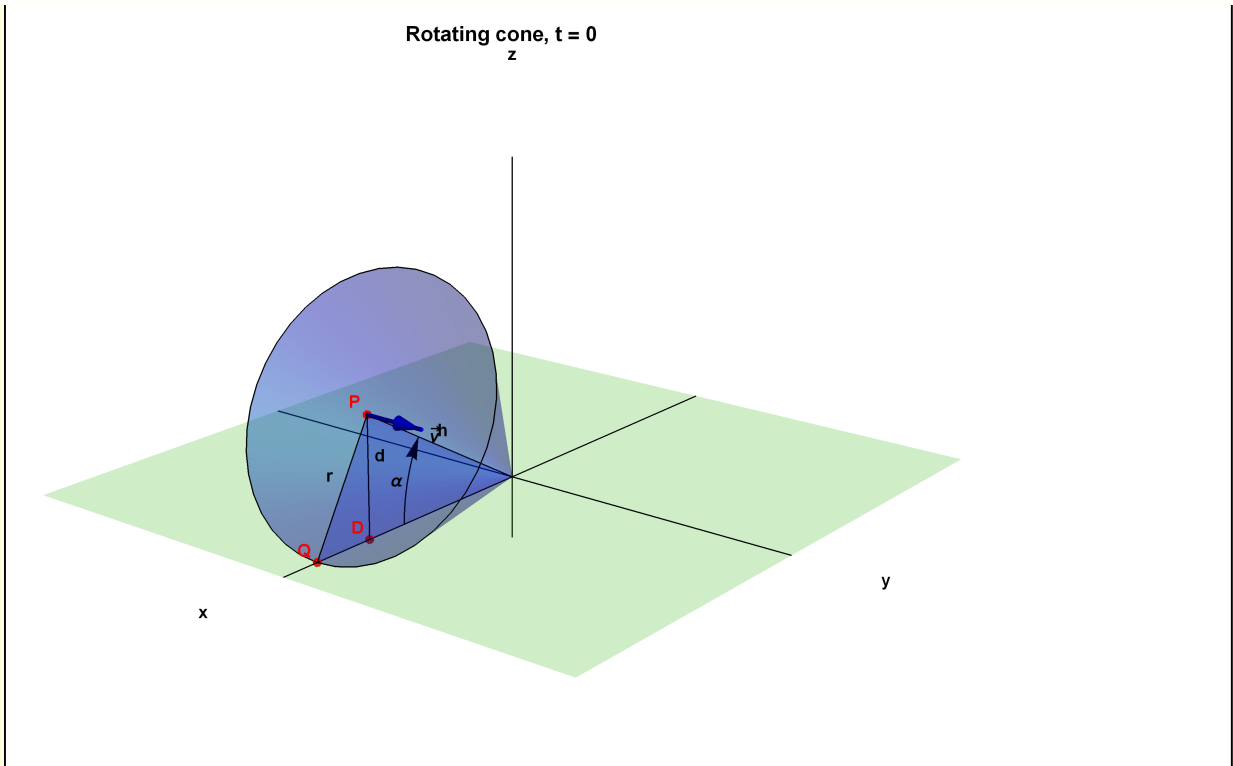
ptPCone = StoC[h,  $\theta$ ,  $\phi$ ];
(* point at contact point of base of cone *)
ptQCone = StoC[ $\sqrt{rCone^2 + h^2}$ ,  $\pi/2$ ,  $\phi$ ];
ptDCone = StoC[dCone,  $\pi/2$ ,  $\phi$ ];
gConeMarkers = {
  {Opacity[0.25, Blue], Specularity[Green, 10], Cone[{ptPCone, 0}, rCone]},
  {Red, PointSize[0.01], {Point[ptPCone], Stl@Text["P", ptPCone + { $\delta$ , 0,  $\delta$ ]}}},
  {Point[ptQCone], Stl@Text["Q", ptQCone + { $\delta$ , 0,  $\delta$ ]}}},
  {Point[ptDCone], Stl@Text["D", ptDCone + { $\delta$ , 0,  $\delta$ ]}}},
  {Black, Line[{0, ptPCone}], Stl@Text["h", Mean[{0, ptPCone}] + {0, 0,  $\delta$ ]}}},
  {Black, Line[{ptQCone, ptPCone}],
   Stl@Text["r", Mean[{ptQCone, ptPCone}] + { $\delta$ , 0,  $\delta$ ]}}},
  {Black, Line[{ptDCone, ptPCone}],
   Stl@Text["d", Mean[{ptDCone, ptPCone}] + {- $\delta$ , 0,  $\delta$ ]}}},
  {Black, Arrowheads[Small], Arrow@Table[StoC[ $\frac{2h}{3}$ ,  $\pi/2 - \xi$ , 0],
    { $\xi$ , 0,  $\alpha$ ,  $\frac{\alpha}{64}$ ]}}},
  {Black, Text[Stl[" $\alpha$ "], StoC[1.1  $\frac{2h}{3}$ ,  $\frac{\pi - \alpha}{2}$ , 0]]}},
  {Opacity[0.25, Blue],
   Polygon[{0, ptQCone, ptPCone}]}}};

vVec = {Blue, Arrowheads[Medium], Arrow@Tube[{ptPCone, ptPCone + {0, 4  $\delta$ , 0}]},
  {Black, Text[Stl[" $\vec{v}$ "], ptPCone + {0, 5  $\delta$ , 0}]}}};
(*gConeMarkers rotated. ptCone is the vector along the cone at time
  t. The cone and all markers are rotated about this axis by  $-\theta$  *)
gConeMarkersRotated = Rotate[gConeMarkers,  $-\theta$ , ptPCone, 0];

lab = Stl@StringForm["Rotating cone, t = ``", NF2@t];
Graphics3D[{gAxes, gPlaneBase, gConeMarkersRotated, vVec},
  ImageSize  $\rightarrow$  500, Axes  $\rightarrow$  False, Boxed  $\rightarrow$  False,
  PlotRange  $\rightarrow$  range, ViewPoint  $\rightarrow$  vp, ViewVertical  $\rightarrow$  {0, 0, 1},
  PlotLabel  $\rightarrow$  lab];

ShowConeInitial[0]

```

```

Clear[Show2DView];
Show2DView[] :=
Module[{0, ex, ey, ez,  $\delta = 0.1$ , h = 1,  $\bar{\phi} = \pi/6$ ,  $\theta = \pi/6$ , xMax = 1.05,
  scale = 1.25, range, gAxes, coneContact, coneContactRotated,  $\bar{\phi}$ Arc,  $\theta$ Arc,
  rVec, rText, rVecRotated, descriptors, lab, G, StoC},
StoC[r_,  $\theta_$ ,  $\phi_$ ] := {r Sin[ $\theta$ ] Cos[ $\phi$ ], r Sin[ $\theta$ ] Sin[ $\phi$ ], r Cos[ $\theta$ ]};
range = {{-1, 1}, {-1, 1}, {-0.2, 1}};
{0, ex, ey, ez} = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
gAxes = {Black, Line[scale{-ex, ex}], Line[scale{-ey, ey}],
  Line[scale{-ez, ez}], Stl@Text["x", (scale +  $\delta$ ) ex],
  Stl@Text["y", (scale +  $\delta$ ) ey], Stl@Text["z", (scale +  $\delta$ ) ez]};

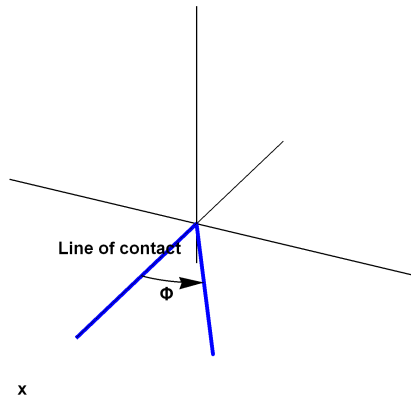
lab = StringForm[" $\bar{\phi}$  measures rotation of line of contact\n about z-axis"];
coneContact = Line[{0, {h, 0, 0}}];
coneContactRotated = Rotate[coneContact, {{1, 0, 0}, {Cos[ $\bar{\phi}$ ], Sin[ $\bar{\phi}$ ], 0}}];
 $\bar{\phi}$ Arc = {Black, Arrowheads[Small], Arrow@Table[StoC[h/2,  $\frac{\pi}{2}$ , x], {x, 0,  $\bar{\phi}$ ,  $\frac{\bar{\phi}}{20}$ }},
  {Black, Text[Stl[" $\bar{\phi}$ "], StoC[1.2  $\frac{h}{2}$ ,  $\frac{\pi}{2}$ ,  $\frac{\bar{\phi}}{2}$ ]]}};
descriptors = {Black, Stl@Text["Line of contact", {0.5, -0.1, 0.1}]};
G[1] = Graphics3D[
  {gAxes, {Blue, Thick, coneContact, coneContactRotated},  $\bar{\phi}$ Arc, descriptors},
  ImageSize  $\rightarrow$  350, Axes  $\rightarrow$  False, Boxed  $\rightarrow$  False,
  PlotRange  $\rightarrow$  range,
  ViewPoint  $\rightarrow$  2 {1, 0.5, 0.5}, ViewVertical  $\rightarrow$  {0, 0, 2},
  PlotLabel  $\rightarrow$  Stl@lab];

descriptors = {Blue, PointSize[0.025], Point[{0, -h}], Point[{0, 0}],
  {Black, Stl@Text["Point of contact", {0, -h 1.1}], Stl@Text["P", {0, 0.1}]}];
rVec = {Black, Line[{{0, 0}, {0, -1}]}];
rText = Text[Stl["r"],  $\frac{1}{2}$  {0, -1} + {- $\delta$ , 0}];
rVecRotated = Rotate[rVec, {{0, 1}, {-Sin[ $\theta$ ], Cos[ $\theta$ ]}];
 $\theta$ Arc =
  {Black, Arrowheads[Small], Arrow@Table[ $\frac{h}{2}$  {Sin[x], -Cos[x]}, {x, 0,  $\theta$ ,  $\frac{\theta}{20}$ }},
  {Black, Text[Stl[" $\theta$ "], 1.2  $\frac{h}{2}$  {Sin[ $\frac{\theta}{2}$ ], -Cos[ $\frac{\theta}{2}$ ]}]}];
lab =
  StringForm[" $\theta$  measures rotation of point of contact \n about axis of cone"];
G[2] = Graphics[{{Circle[], rVec, rText, rVecRotated,  $\theta$ Arc, descriptors},
  ImageSize  $\rightarrow$  200, PlotLabel  $\rightarrow$  Stl@lab];
Grid[{{G[1], G[2]}}];

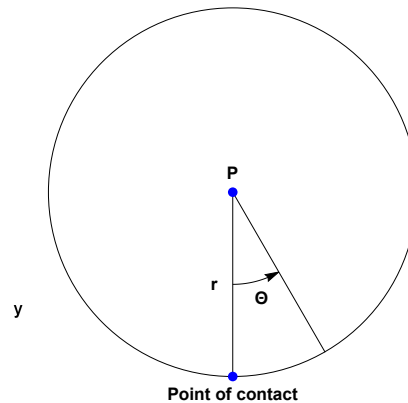
Show2DView[]

```

Φ measures rotation of line of contact
about z-axis



Θ measures rotation of point of contact
about axis of cone



```

Clear[ShowConeSolution1];
ShowConeSolution1[t_] :=
Module[{scale = 1.25, δ = 0.1, v = 1, h = 1, α = π/9,
  0, ex, ey, ez, gAxes, range, gPlaneBase, θ, φ, θ̄, θ̄0, θ̄0, θ̄0,
  rCone, gCone, ptPCone, ptQCone, ptVbase, ptVtip, QRotationVector,
  gConeMarkers, gConeMarkersRotated, axisCone, vVec, ωVec, lab, StoC},
QRotationVector[vList_, size_, radius_] :=
Module[{g = Graphics3D@Cylinder[{{0, 0, 0}, {size, 0, 0}}, radius]},
  {Arrowheads[{{Automatic, Automatic, g}}, Arrow[Tube[vList]]]};
StoC[r_, θ_, φ_] := {r Sin[θ] Cos[φ], r Sin[θ] Sin[φ], r Cos[θ]};
range = {{-1, 1}, {-1, 1}, {-1, 1}};
{0, ex, ey, ez} = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
gAxes = {Black, Line[scale {-ex, ex}], Line[scale {-ey, ey}],
  Line[scale {-ez, ez}], Stl@Text["x", (scale + δ) ex],
  Stl@Text["y", (scale + δ) ey], Stl@Text["z", (scale + δ) ez]};
gPlaneBase =
  {Opacity[0.25, Lighter[Green, 0.25]],
  Polygon[scale {{-1, -1, 0}, {1, -1, 0}, {1, 1, 0}, {-1, 1, 0}}]};

(* initial angle between z-axis and rotation axis of the cone *)
θ = π/2 - α;
θ̄0 = 0; (* cone is initially oriented along x-axis *)
θ̄0 = 0; (* initial line between center of cone and point of contact *)
θ = θ̄0 +  $\frac{v}{h \tan[\alpha]}$  t;
θ̄ = θ̄0 + Sin[α] θ;
φ = θ̄;
rCone = h Tan[α];
(* point at center of base of cone *)
ptPCone = StoC[h, θ, φ];

```

```

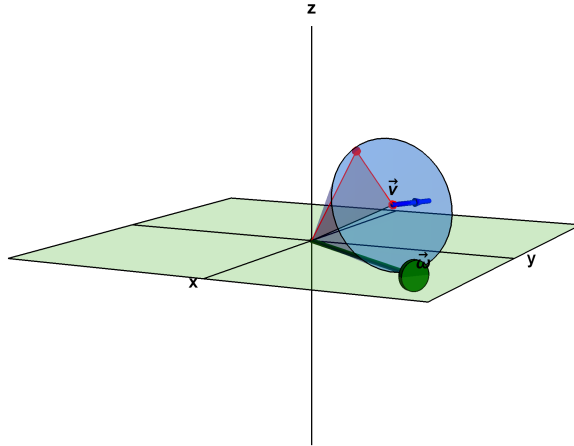
(* point at contact point of base of cone *)
ptQCone = StoC[h,  $\pi/2$ ,  $\phi$ ];
ptVbase = ptPCone;
ptVtip = StoC[h,  $\theta$ ,  $\phi + \pi/12$ ];

gConeMarkers =
  {{Opacity[0.25, Blue], Specularity[Green, 10], Cone[{ptPCone, 0}, rCone]},
   {Red, PointSize[0.01], Point[ptPCone], Point[ptQCone],
    Line[{0, ptQCone}], Line[{ptQCone, ptPCone}],
    Opacity[0.25, Red], Polygon[{0, ptQCone, ptPCone}]}}};
(*gConeMarkers rotated. ptCone is the vector along the cone at time
  t. The cone and all markers are rotated about this axis by - $\theta$  *)
gConeMarkersRotated = Rotate[gConeMarkers, - $\theta$ , ptPCone, 0];
vVec = {Blue, Arrowheads[Small], Arrow@Tube[{ptVbase, ptVtip}],
  {Black, Text[St1[" $\vec{v}$ "], ptVbase + {0, 0,  $\delta$ ]}]};
 $\omega$ Vec = {Directive[Darker[Green, 0.5]], QRotationVector[{0, 1.15 ptQCone},
  0.1, 0.3], {Black, Text[St1[" $\vec{\omega}$ "], 1.25 ptQCone + {0, 0,  $\delta$ ]}]};

lab = Stl@StringForm["Rotating cone, t = ``", NF2@t];
Graphics3D[{gAxes, gPlaneBase, gConeMarkersRotated, vVec,  $\omega$ Vec},
  ImageSize  $\rightarrow$  500, Axes  $\rightarrow$  False, Boxed  $\rightarrow$  False,
  PlotRange  $\rightarrow$  {{-2, 2}, {-2, 2}, {-2, 2}},
  ViewPoint  $\rightarrow$  2 {1, 0.5, 0.2}, ViewVertical  $\rightarrow$  {0, 0, 1},
  PlotLabel  $\rightarrow$  lab];

```

```
ShowConeSolution1[1]
```

Rotating cone, $t = 1.00$ 

```

Clear[ShowConeSolution2];
ShowConeSolution2[t_] :=
Module[{scale = 1.25,  $\delta = 0.1$ ,  $v = 1$ ,  $h = 1$ ,  $\alpha = \pi/9$ , 0, ex, ey, ez, gAxes,
  range, gPlaneBase,  $\theta$ ,  $\phi$ ,  $\theta$ ,  $\bar{\theta}$ ,  $\theta\theta$ ,  $\bar{\theta}\theta$ , rCone, gCone, ptPCone, ptQCone,
  ptVbase, ptVtip, pt $\omega$ 12tip, pt $\omega$ 23tip, QRotationVector, gConeMarkers,
  gConeMarkersRotated, axisCone, vVec,  $\omega$ 13Vec,  $\omega$ 12Vec,  $\omega$ 23Vec, lab, StoC},
  QRotationVector[vList_, size_, radius_] :=
  Module[{g = Graphics3D@Cylinder[{{0, 0, 0}, {size, 0, 0}}, radius]},
    {Arrowheads[{{Automatic, Automatic, g}}, Arrow[Tube[vList]]]};
  StoC[r_,  $\theta$ _,  $\phi$ _] := {r Sin[ $\theta$ ] Cos[ $\phi$ ], r Sin[ $\theta$ ] Sin[ $\phi$ ], r Cos[ $\theta$ ]};
  range = {{-1, 1}, {-1, 1}, {-1, 1}};
  {0, ex, ey, ez} = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
  gAxes = {Black, Line[scale {-ex, ex}], Line[scale {-ey, ey}],
    Line[scale {-ez, ez}], St1@Text["x", (scale +  $\delta$ ) ex],
    St1@Text["y", (scale +  $\delta$ ) ey], St1@Text["z", (scale +  $\delta$ ) ez]};
  gPlaneBase =
  {Opacity[0.25, Lighter[Green, 0.25]],
    Polygon[scale {{-1, -1, 0}, {1, -1, 0}, {1, 1, 0}, {-1, 1, 0}}]};

  (* initial angle between z-axis and rotation axis of the cone *)

```

```

 $\theta = \pi/2 - \alpha;$ 
 $\bar{x}\theta = 0;$  (* cone is initially oriented along x-axis *)
 $\theta\theta = 0;$  (* initial line between center of cone and point of contact *)
 $\theta = \theta\theta + \frac{v}{h \tan[\alpha]} t;$ 
 $\bar{x} = \bar{x}\theta + \sin[\alpha] \theta;$ 
 $\phi = \bar{x};$ 
rCone = h Tan[ $\alpha$ ];
(* point at center of base of cone *)
ptPCone = StoC[h,  $\theta$ ,  $\phi$ ];
(* point at contact point of base of cone *)
ptQCone = StoC[h,  $\pi/2$ ,  $\phi$ ];
ptVbase = ptPCone;
ptVtip = StoC[h,  $\theta$ ,  $\phi + \pi/12$ ];
pt $\omega$ 12tip = StoC[h,  $\pi/2 - \alpha$ ,  $\phi$ ];
pt $\omega$ 23tip = StoC[h,  $\pi - \alpha$ ,  $\phi$ ];

gConeMarkers =
{{Opacity[0.25, Blue], Specularity[Green, 10], Cone[{ptPCone, 0}, rCone]},
 {Red, PointSize[0.01], Point[ptPCone], Point[ptQCone],
  Line[{0, ptQCone}], Line[{ptQCone, ptPCone}],
  Opacity[0.25, Red], Polygon[{0, ptQCone, ptPCone]}}};
(*gConeMarkers rotated. ptCone is the vector along the cone at time
 t. The cone and all markers are rotated about this axis by  $-\theta$  *)
gConeMarkersRotated = Rotate[gConeMarkers,  $-\theta$ , ptPCone, 0];
vVec = {Blue, Arrowheads[Small], Arrow@Tube[{ptVbase, ptVtip}],
 {Black, Text[St1[" $\vec{v}$ "], ptVbase + {0, 0,  $\delta$ ]}]};
 $\omega$ 13Vec = {Directive[Darker[Green, 0.5]], QRotationVector[{0, 1.15 ptQCone},
 0.1, 0.3], {Black, Text[St1[" $\vec{\omega}_{1,3}$ "], 1.35 ptQCone + {0, 0,  $\delta$ ]}]};
 $\omega$ 12Vec = {Red, QRotationVector[{0, 1.15 pt $\omega$ 12tip}, 0.1, 0.3],
 {Black, Text[St1[" $\vec{\omega}_{1,2}$ "], 1.35 pt $\omega$ 12tip + {0, 0,  $\delta$ ]}]};
 $\omega$ 23Vec = {Red, QRotationVector[{0, 1.15 pt $\omega$ 23tip}, 0.1, 0.3],
 {Black, Text[St1[" $\vec{\omega}_{2,3}$ "], 1.35 pt $\omega$ 23tip + {0, 0,  $\delta$ ]}]};

lab = St1@StringForm["Rotating cone, t = ``", NF2@t];
Graphics3D[{gAxes, gPlaneBase, gConeMarkersRotated, vVec,  $\omega$ 12Vec,  $\omega$ 23Vec,  $\omega$ 13Vec},
 ImageSize  $\rightarrow$  500, Axes  $\rightarrow$  False, Boxed  $\rightarrow$  False,
 PlotRange  $\rightarrow$  {{-2, 2}, {-2, 2}, {-2, 2}},
 ViewPoint  $\rightarrow$  2 {1, 0.5, 0.2}, ViewVertical  $\rightarrow$  {0, 0, 1},
 PlotLabel  $\rightarrow$  lab];

ShowConeSolution2[0.0]

```

Rotating cone, $t = 0$

