Mechanics - Morin 9.5 Nice Cylinder 10-29-18

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Initialization: Be sure the file *NTGUtilityFunctions.m* is in the same directory as that from which this notebook was loaded. Then execute the cell immediately below by mousing left on the cell bar to the right of that cell and then typing "shift" + "enter". Respond "Yes" in response to the query to evaluate initialization cells.

```
SetDirectory[NotebookDirectory[]];
 (* set directory where source files are located *)
Get["NTGUtilityFunctions.m"]; (* Load utilities package *)
```

Background

This is the third in a series of Mathematica notebooks on classical mechanics. This series was motivated by a close reading and problem solving project I undertook in 2014. The focus of my attention was the text ***Introduction to Classical Mechanics with Problems and Solutions***, by David Morin. This is a good book from which to learn and has a great collection of problems. I purchased it and recommend that those with interests in this topic acquire it for their library. I do note that an earlier version can be found on the web. This year, when I returned to this project, I decided to focus on generating Mathematica notebooks on material covered in *Chapter 9 Angular Momentum, Part II (General $\vec{L}$),* which deals the 3-D rigid body dynamics. This topic is notorious difficult/confusing for students and I felt I just skimmed by as a graduate student. I return in retirement after all these years to pay my dues and really understand how to solve problems is this area.

Although Morin's Chapter 9 guides these notebooks, I made frequent use of other sources such as textbooks available in libraries or on the web. I also found lots of video lectures available on YouTube. Confused about some physics topics? Google it and you'll be amazed what you find. Some relevant texts are

Classical Mechanics, Hebert Goldstein (my original text at University, late 60s). Newer versions exist.
Mechanics: Volume1 A Course in Theoretical Physics, L. D. Landau and E. M. Lifshitz.
Classical Mechanics, John. R. Taylor
Classical Mechanics of Particles and Systems, Stephen T. Thornton, Jerry B. Marion
Analytical Mechanics, G. R. Fowles, G. L. Cassiday

Analytical Mechanics, Louis N. Hand, Janet D. Finch

I find Mathematica useful for this topic. It facilitates calculations, provides a vehicle for creating instructive visualizations and allows one to quickly generate numerical solutions. Mathematica is a favorite tool of mine but I think it is crucially important to also work with pen and paper. Our brains are closely linked to our hands and one thinks differently with a pen in hand than when sitting before a computer screen. For serious thoughts on this, read *The Craftsman*, by Richard Sennett.

Nice cylinder

A nice cylinder is one for which the moment of inertia is the same for each of the three axes of rotation through its center of mass.

There are several ways I can use Mathematica to calculate the moment of inertia of a cylinder. For convenience, I define the inertial tensor

```
I = {{y² + z², -x y, -x z},
     {-x y, x² + z², -y z},
     {-x z, -y z, x² + y²}};
I // MatrixForm
```

$$\begin{pmatrix} y^2 + z^2 & -x\,y & -x\,z \\ -x\,y & x^2 + z^2 & -y\,z \\ -x\,z & -y\,z & x^2 + y^2 \end{pmatrix}$$

I can perform a general integration over all space, restricting the range of integration with Booleans.

```
w[1] = With[{ρ = M/(π R² h)},

   ρ Integrate[
     I Boole[x² + y² < R²] Boole[-h/2 < z < h/2],
     {x, -∞, ∞}, {y, -∞, ∞}, {z, -∞, ∞}, Assumptions → R > 0 && h > 0]] // Expand
```

$$\left\{\left\{\frac{h^2 M}{12} + \frac{M R^2}{4}, 0, 0\right\}, \left\{0, \frac{h^2 M}{12} + \frac{M R^2}{4}, 0\right\}, \left\{0, 0, \frac{M R^2}{2}\right\}\right\}$$

I can transform to cylindrical coordinates and perform the appropriate definite integrals

```
tranformationRules =
  Thread[Rule[{x, y, z}, CoordinateTransform["Cylindrical" → "Cartesian", {r, θ, z}]]]
```

$$\{x \to r\,\text{Cos}[\theta], \, y \to r\,\text{Sin}[\theta], \, z \to z\}$$

```
Icyl = I /. tranformationRules // Simplify
```

$$\{\{z^2 + r^2 \sin[\theta]^2, \ -r^2 \cos[\theta] \sin[\theta], \ -r z \cos[\theta]\},$$
$$\{-r^2 \cos[\theta] \sin[\theta], \ z^2 + r^2 \cos[\theta]^2, \ -r z \sin[\theta]\}, \ \{-r z \cos[\theta], \ -r z \sin[\theta], \ r^2\}\}$$

```
w[2] = Module[{ρ = M/(π R² h)},
   ρ Integrate[r Icyl, {r, 0, R}, {θ, 0, 2π}, {z, -h/2, h/2}]] // Expand
```

$$\left\{\left\{\frac{h^2 M}{12} + \frac{M R^2}{4}, \ 0, \ 0\right\}, \ \left\{0, \ \frac{h^2 M}{12} + \frac{M R^2}{4}, \ 0\right\}, \ \left\{0, \ 0, \ \frac{M R^2}{2}\right\}\right\}$$

I can take advantage of *Region* capabilities.

```
w[3] = Module[{ρ = M/(π R² h), R},
   R = ImplicitRegion[x² + y² ≤ R² && -h/2 < z < h/2, {x, y, z}];
   ρ Integrate[I, {x, y, z} ∈ R, Assumptions → {R > 0, h > 0}]] // Expand
```

$$\left\{\left\{\frac{h^2 M}{12} + \frac{M R^2}{4}, \ 0, \ 0\right\}, \ \left\{0, \ \frac{h^2 M}{12} + \frac{M R^2}{4}, \ 0\right\}, \ \left\{0, \ 0, \ \frac{M R^2}{2}\right\}\right\}$$

Or, even take advantage that *Cylinder* is a known entity,

```
TEST
```

$$\{\{RegionDimension, 3\}, \{RegionMeasure, h \pi R^2\},$$
$$\{RegionCentroid, \{0, 0, 0\}\}, \{Volume, h \pi R^2\}, \{MomentOfInertia,$$
$$\left\{\left\{\frac{1}{12} h^3 \pi R^2 + \frac{1}{4} h \pi R^4, \ 0, \ 0\right\}, \ \left\{0, \ \frac{1}{12} h^3 \pi R^2 + \frac{1}{4} h \pi R^4, \ 0\right\}, \ \left\{0, \ 0, \ \frac{1}{2} h \pi R^4\right\}\right\}\}\}$$

```
MapAt[(Expand@Simplify[# M/(π R² h)]) &, TEST, {5, 2}]
```

$$\{\{RegionDimension, 3\}, \{RegionMeasure, h \pi R^2\},$$
$$\{RegionCentroid, \{0, 0, 0\}\}, \{Volume, h \pi R^2\},$$
$$\{MomentOfInertia, \left\{\left\{\frac{h^2 M}{12} + \frac{M R^2}{4}, \ 0, \ 0\right\}, \ \left\{0, \ \frac{h^2 M}{12} + \frac{M R^2}{4}, \ 0\right\}, \ \left\{0, \ 0, \ \frac{M R^2}{2}\right\}\right\}\}\}$$

```
Module[{ρ = M/(π R² h), object, measures, calculations, results},
 object = Cylinder[{{0, 0, -h/2}, {0, 0, h/2}}, R];
 measures =
  {RegionDimension, RegionMeasure, RegionCentroid, Volume, MomentOfInertia};
 calculations = Through[measures[object]];
 results = Transpose[{measures, calculations}];
 results = Simplify[results, {{x, y, z} ∈ Reals, R > 0, h > 0}] // Expand;
 results = MapAt[(Expand@Simplify[# ρ]) &, results, {5, 2}];
 Grid[results, Frame → All]]
```

| RegionDimension	3
RegionMeasure	$h \pi R^2$
RegionCentroid	{0, 0, 0}
Volume	$h \pi R^2$
MomentOfInertia	$\left\{\left\{\frac{h^2 M}{12} + \frac{M R^2}{4}, 0, 0\right\}, \left\{0, \frac{h^2 M}{12} + \frac{M R^2}{4}, 0\right\}, \left\{0, 0, \frac{M R^2}{2}\right\}\right\}$

Equate $\mathcal{I}_{xx}$ with $\mathcal{I}_{zz}$ to determine the value of h that makes a cylinder "nice."

```
Solve[(M R²)/2 == M (h²/12 + R²/4), h][[2, 1]]
```

$h \to \sqrt{3}\ R$

I illustrate the rotations of a nice cylinder about its three principle axes.

```
Module[{ζMax = 2 π, dζ = 2 π/36, frames},
 frames = Table[ShowNiceCylinder[ζ], {ζ, 0, ζMax, dζ}];
 ListAnimate[frames]]
```
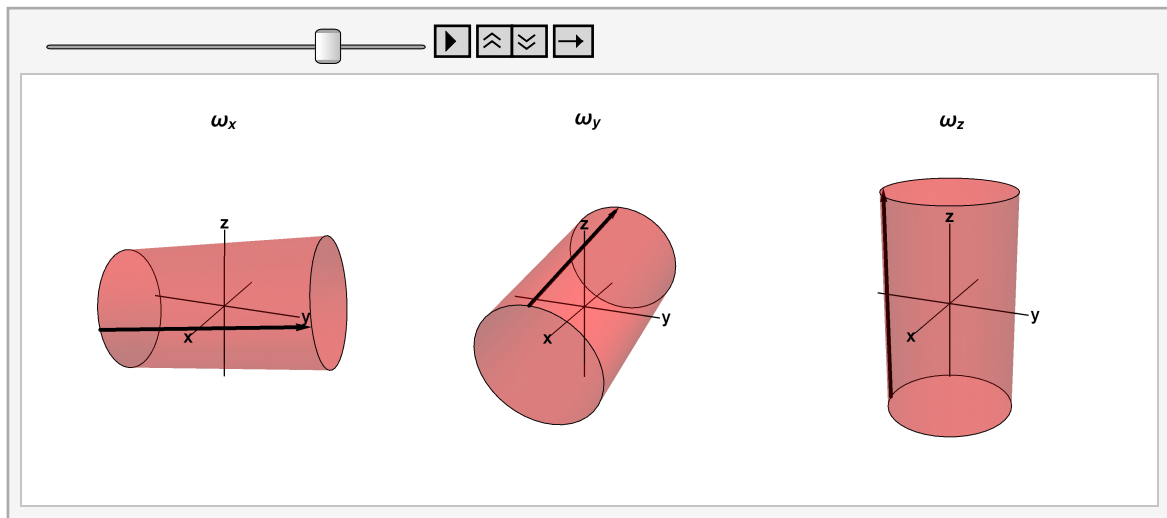
```
Clear[ShowNiceCylinder];
ShowNiceCylinder[ξ_] :=
 Module[{scale = 1.25, δ = 0.1, viewPoint = {2.4, 1, 1}, cylinder, gCylinder,
   gCylinderZ, gCylinderX, gCylinderY, O, ex, ey, ez, axes, markerLine, StoC, G},
  {O, ex, ey, ez} = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
  StoC[r_, θ_, ϕ_] := {r Cos[ϕ] Sin[θ], r Sin[θ] Sin[ϕ], r Cos[θ]};
  axes = {Black, Line[scale {-ex, ex}], Line[scale { -ey, ey}],
    Line[scale {- ez, ez}], Stl@Text["x", (scale + δ) ex],
    Stl@ Text["y", (scale + δ) ey], Stl@Text["z", (scale + δ) ez]};

  {cylinder, markerLine} = With[{R = 1, h = √3 , θ = π/2, dϕ = 2 π/64 },
    {Cylinder[{{0, 0, -h}, {0, 0, h}}, R],
     markerLine = Arrow[{{R, 0, -h}, {R, 0, h}}]}] ;
  gCylinder =
   {{Red, Opacity[0.3], cylinder}, {Directive[Black, Thick], markerLine}};
  gCylinderZ = Rotate[gCylinder, ξ, ez, O];
  gCylinderY = Rotate[gCylinder, ξ, ey, O];
  gCylinderX = Rotate[gCylinder, ξ, ex, O];

  G[1] = Graphics3D[{axes, gCylinderX}, Boxed → False,
    ViewPoint → viewPoint, PlotLabel → Stl["ωₓ"], SphericalRegion → True];
  G[2] = Graphics3D[{axes, gCylinderY}, Boxed → False, ViewPoint → viewPoint,
    PlotLabel → Stl["ω_y"], SphericalRegion → True];
  G[3] = Graphics3D[{axes, gCylinderZ}, Boxed → False, ViewPoint → viewPoint,
    PlotLabel → Stl["ω_z"], SphericalRegion → True];
  Grid[{{G[1], G[2], G[3]}}]]
```