
Inverted Pendulum - Morin 6.5

07-16-16

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Initialization: Be sure the files *NTGStylesheet2.nb* and *NTGUtilityFunctions.m* is are in the same directory as that from which this notebook was loaded. Then execute the cell immediately below by mousing left on the cell bar to the right of that cell and then typing “shift” + “enter”. Respond “Yes” in response to the query to evaluate initialization cells.

In[1]:=

```
SetDirectory[NotebookDirectory[]];  
(* set directory where source files are located *)  
SetOptions[EvaluationNotebook[], (* load the StyleSheet *)  
  StyleDefinitions -> Get["NTGStylesheet2.nb"]];  
Get["NTGUtilityFunctions.m"]; (* Load utilities package *)
```

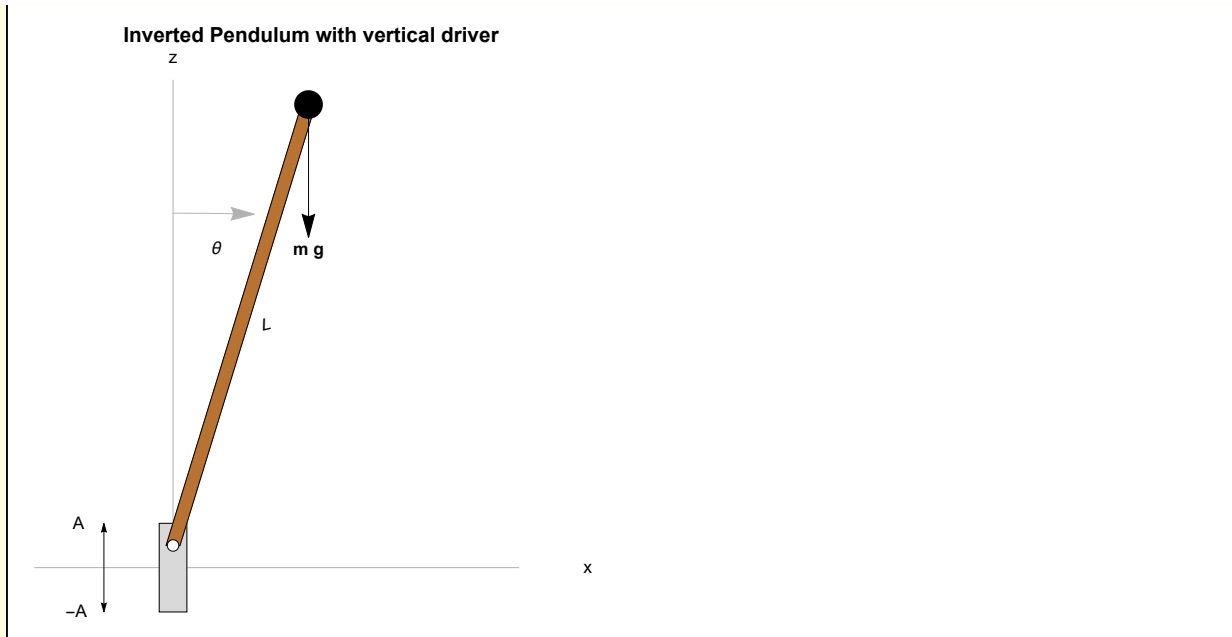
Purpose

I work through Morin's treatment of the inverted pendulum — *Introduction to Classical Mechanics*, David Morin, Problem 6.5.

This is also problem 67 of the Harvard Problem of the Week series.

<http://webdev.physics.harvard.edu/academics/undergrad/problems.html>

The inverted pendulum is a classic problem with a long history and large literature. The variant discussed here involves preventing the fall of a pendulum by applying a high frequency vertical driving motion to the base of the pendulum.



Initialization

To facilitate the use of subscript notation I load the Notation package and “symbolize” some subscript structures that will appear below.

In[7]:=

```
Needs["Notation`"]
```

In[8]:=

```
Symbolize[Aθ];
Symbolize[ωd];
Symbolize[ωp];
Symbolize[Cθ];
Symbolize[C1];
Symbolize[θθ]; Symbolize[τslow];
```

Derivation of equations of motion

An ideal pendulum is considered, with mass concentrated at the tip. The location of the tip is defined by

In[9]:=

```
w1[1] = {x[t] -> L Sin[θ[t]], z[t] -> A[t] + L Cos[θ[t]]}
```

Out[9]=

```
{x[t] -> L Sin[θ[t]], z[t] -> A[t] + L Cos[θ[t]]}
```

where $A(t)$ is the amplitude of the vertical driver.

The Lagrangian formalism is used to derive the equation of motion. The kinetic energy is

$$\text{In[10]:= } \mathbf{w1[2]} = T[t] \rightarrow \frac{m}{2} (D[x[t], t]^2 + D[z[t], t]^2)$$

$$\text{Out[10]= } T[t] \rightarrow \frac{1}{2} m (x'[t]^2 + z'[t]^2)$$

or, in terms of a polar angle,

$$\text{In[11]:= } \mathbf{w1[3]} = \mathbf{w1[2]} /. \text{Map}[D[\#, t] \&, \mathbf{w1[1]}, \{2\}]$$

$$\text{Out[11]= } T[t] \rightarrow \frac{1}{2} m (L^2 \text{Cos}[\theta[t]]^2 \theta'[t]^2 + (A'[t] - L \text{Sin}[\theta[t]] \theta'[t])^2)$$

The potential energy is

$$\text{In[12]:= } \mathbf{w1[4]} = V[t] \rightarrow m g z[t]$$

$$\text{Out[12]= } V[t] \rightarrow g m z[t]$$

$$\text{In[13]:= } \mathbf{w1[5]} = \mathbf{w1[4]} /. \mathbf{w1[1]}$$

$$\text{Out[13]= } V[t] \rightarrow g m (A[t] + L \text{Cos}[\theta[t]])$$

The Lagrangian is

$$\text{In[14]:= } \mathbf{w1[6]} = \mathcal{L}[t] \rightarrow T[t] - V[t]$$

$$\text{Out[14]= } \mathcal{L}[t] \rightarrow T[t] - V[t]$$

In detail

$$\text{In[15]:= } \mathbf{w1[7]} = \mathbf{w1[6]} /. \mathbf{w1[3]} /. \mathbf{w1[5]} // \text{Simplify}[\#, \text{Trig} \rightarrow \text{True}] \&$$

$$\text{Out[15]= } \mathcal{L}[t] \rightarrow \frac{1}{2} m (-2 g A[t] + A'[t]^2 - 2 L \text{Sin}[\theta[t]] A'[t] \theta'[t] + L (-2 g \text{Cos}[\theta[t]] + L \theta'[t]^2))$$

The Euler Lagrange equation is

$$\text{In[16]:= } \mathbf{w1[8]} = (D[D[\#, \theta'[t]], t] == D[\#, \theta[t]]) \& [\mathcal{L}[t] /. \mathbf{w1[7]}]$$

$$\text{Out[16]= } \frac{1}{2} m (-2 L \text{Cos}[\theta[t]] A'[t] \theta'[t] - 2 L \text{Sin}[\theta[t]] A''[t] + 2 L^2 \theta''[t]) == \frac{1}{2} m (2 g L \text{Sin}[\theta[t]] - 2 L \text{Cos}[\theta[t]] A'[t] \theta'[t])$$

Upon specifying an explicit form for A(t)

In[17]:= **w1[9] = w1[8] /. A -> (A0 Cos[ωd #]) & // RE**

Out[17]=
$$\frac{1}{2} m (2 A_0 L \omega_d^2 \cos[t \omega_d] \sin[\theta[t]] + 2 A_0 L \omega_d \cos[\theta[t]] \sin[t \omega_d] \theta'[t] + 2 L^2 \theta''[t]) ==$$

$$\frac{1}{2} m (2 g L \sin[\theta[t]] + 2 A_0 L \omega_d \cos[\theta[t]] \sin[t \omega_d] \theta'[t])$$

In[18]:= **w1[10] = Solve[w1[9], θ''[t]] [[1, 1]] // Simplify // RE**

Out[18]=
$$\theta''[t] == \frac{(g - A_0 \omega_d^2 \cos[t \omega_d]) \sin[\theta[t]]}{L}$$

It is more convenient to work with a dimensionless form of this equation. Introduce the classic oscillation frequency for a pendulum

In[19]:= **def[ωp] = ωp == √g/L**

Out[19]=
$$\omega_p == \sqrt{\frac{g}{L}}$$

In[21]:= **w1[11] = w1[10] /. Sol[def[ωp], g]**

Out[21]=
$$\theta''[t] == \frac{(L \omega_p^2 - A_0 \omega_d^2 \cos[t \omega_d]) \sin[\theta[t]]}{L}$$

Two dimensionless parameters suggest themselves — the ratio of the maximum amplitude of the high frequency driver A_0 to the pendulum length L , and the ratio of driver frequency to the characteristic pendulum frequency

In[22]:= **{def[a] = a == A0/L, def[f] = f == ωd/ωp}**

Out[22]=
$$\left\{ a == \frac{A_0}{L}, f == \frac{\omega_d}{\omega_p} \right\}$$

In[24]:= **w1[12] = w1[11] /. Sol[def[a], A0] /. Sol[def[f], ωp] // Simplify**

Out[24]=
$$\theta''[t] == - \frac{\omega_d^2 (-1 + a f^2 \cos[t \omega_d]) \sin[\theta[t]]}{f^2}$$

Introduce a characteristic time scale. For this high frequency phenomenon the natural choice is $1/\omega_d$

In[25]:= **def[τ] = τ == ωd t**

Out[25]=
$$\tau == t \omega_d$$

In[26]:= **w1[13] = w1[12] /. $\theta \rightarrow (\theta[\#\omega_d] L) \&$ /. t $\rightarrow \tau / \omega_d$ // RE**

Out[26]=
$$L \omega_d^2 \theta''[\tau] == - \frac{\omega_d^2 (-1 + a f^2 \text{Cos}[\tau]) \text{Sin}[L \theta[\tau]]}{f^2}$$

and finally, an equation of motion in a form suitable for analysis

In[27]:= **w1[14] = MapEqn[(# / (L ω_d^2)) &, w1[13]]**

Out[27]=
$$\theta''[\tau] == - \frac{(-1 + a f^2 \text{Cos}[\tau]) \text{Sin}[L \theta[\tau]]}{f^2 L}$$

This equation of motion becomes analytically tractable for small amplitude oscillations

In[28]:= **w1[15] = w1[14] /. Sin[x_] \rightarrow x // Expand**

Out[28]=
$$\theta''[\tau] == \frac{\theta[\tau]}{f^2} - a \text{Cos}[\tau] \theta[\tau]$$

This is actually Mathieu's equation and Mathematica knows the solution. For some nominal initial conditions

In[29]:= **w1[16] = DSolve[{w1[15], $\theta'[\theta] == 0$, $\theta[\theta] == \theta_0$ }, $\theta[\tau]$, τ] [1, 1]**

Out[29]=
$$\theta[\tau] \rightarrow \frac{\theta_0 \text{MathieuC}\left[-\frac{4}{f^2}, -2a, \frac{\tau}{2}\right]}{\text{MathieuC}\left[-\frac{4}{f^2}, -2a, \theta\right]}$$

If the frequency of the driver is lower than or on the order of the natural frequency of the pendulum it can't possibly stop the pendulum from falling — so the regime of interest for an inverted pendulum is $f \gg 1$. Also, consistent with the original diagram the vertical extent of the driver is much less than the length of the pendulum and so $a \ll 1$.

Some nominal parameters

Nominal parameters						
ω_p s ⁻¹	ω_d s ⁻¹	A_0 m	L m	a	f	$a f^2$
$\sqrt{10}$	10	0.1	1	0.1	$\sqrt{10}$	1.
$\sqrt{10}$	100	0.1	1	0.1	$10\sqrt{10}$	100.

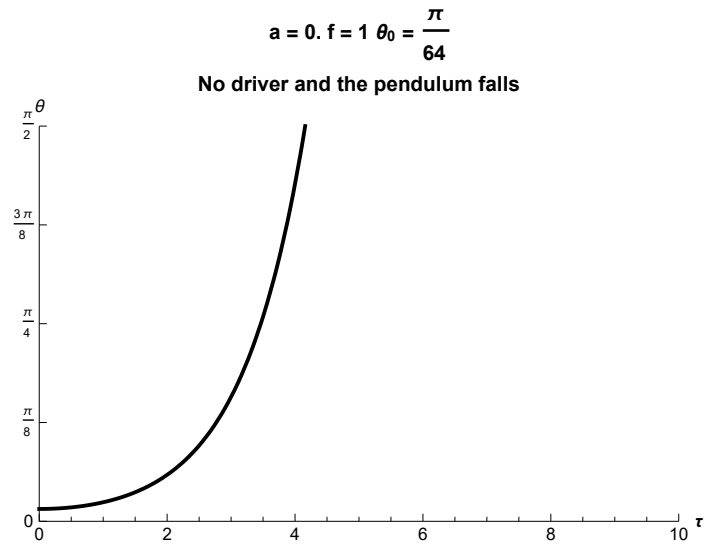
2 Visualization

Before proceeding to analysis of the equation of motion, it is instructive to visually examine the characteristic behaviors of the driven inverted pendulum. The initial conditions are that the pendulum is released from rest slightly off vertical

In[30]:=

```
Module[{a = 0.0, f = 1,  $\theta_0 = \pi / 64$ ,  $\tau_{\text{Max}} = 10$ ,  $\theta_{\text{Exact}}$ , soln, lab},
 $\theta_{\text{Exact}}[\tau_] := \frac{\theta_0 \text{MathieuC}[-\frac{4}{f^2}, -2 a, \frac{\tau}{2}]}{\text{MathieuC}[-\frac{4}{f^2}, -2 a, 0]}$ ;
lab = Stl@
StringForm["a = `` f = ``  $\theta_0 = ``\n$ No driver and the pendulum falls", a, f,  $\theta_0$ ];
Plot[ $\theta_{\text{Exact}}[\tau]$ , { $\tau$ , 0,  $\tau_{\text{Max}}$ }, PlotStyle -> BLACK,
PlotLabel -> Stl[lab],
AxesLabel -> {Stl[" $\tau$ "], " $\theta$ "}, PlotRange -> {{0,  $\tau_{\text{Max}}$ }, {0,  $\pi/2$ }},
Ticks -> {Automatic, {0,  $\pi/8$ ,  $\pi/4$ ,  $3\pi/8$ ,  $\pi/2$ }}]
```

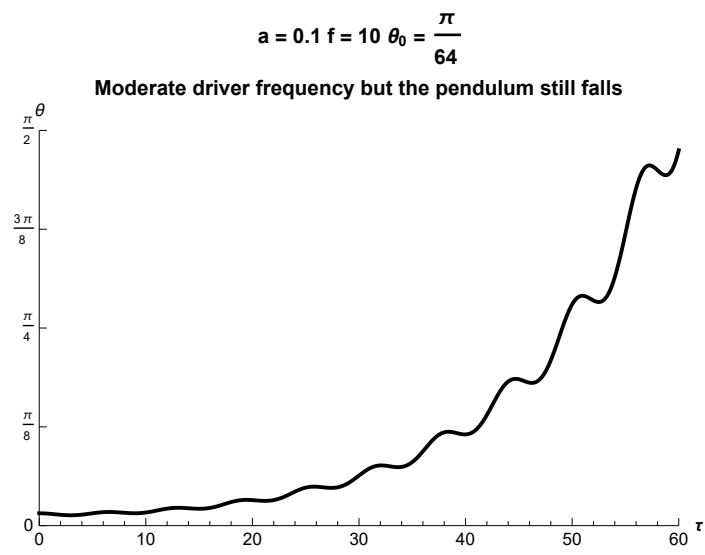
Out[30]=



In[31]=

```
Module[{a = 0.1, f = 10,  $\theta_0 = \pi/64$ ,  $\tau_{\text{Max}} = 60$ ,  $\theta_{\text{Exact}}$ , soln, lab},
 $\theta_{\text{Exact}}[\tau_] := \frac{\theta_0 \text{MathieuC}[-\frac{4}{f^2}, -2 a, \frac{\tau}{2}]}{\text{MathieuC}[-\frac{4}{f^2}, -2 a, 0]}$ ;
lab = Stl@StringForm[
  "a = `` f = ``  $\theta_0 = ``$ \nModerate driver frequency but the pendulum still falls",
  a, f,  $\theta_0$ ];
Plot[ $\theta_{\text{Exact}}[\tau]$ , { $\tau$ , 0,  $\tau_{\text{Max}}$ }, PlotStyle -> BLACK,
  PlotLabel -> Stl[lab],
  AxesLabel -> {Stl[" $\tau$ "], " $\theta$ "}, PlotRange -> {{0,  $\tau_{\text{Max}}$ }, {0,  $\pi/2$ }},
  Ticks -> {Automatic, {0,  $\pi/8$ ,  $\pi/4$ ,  $3\pi/8$ ,  $\pi/2$ }}]
```

Out[31]=

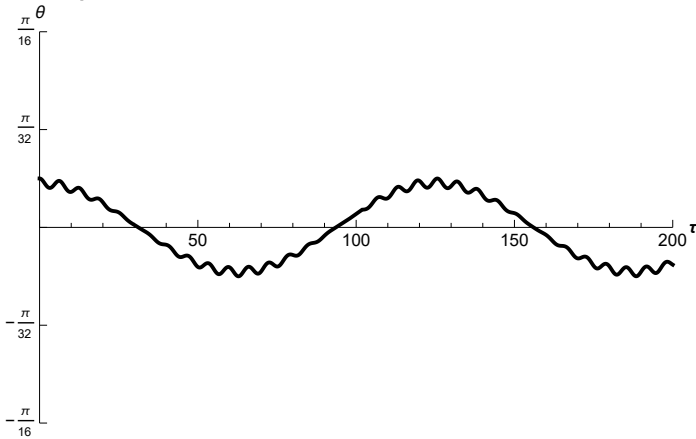


In[32]:=

```
Module[{a = 0.1, f = 20,  $\theta_0 = \pi/64$ ,  $\tau_{\text{Max}} = 200$ ,  $\theta_{\text{Exact}}$ , lab},
 $\theta_{\text{Exact}}[\tau_] := \frac{\theta_0 \text{MathieuC}[-\frac{4}{f^2}, -2a, \frac{\tau}{2}]}{\text{MathieuC}[-\frac{4}{f^2}, -2a, 0]}$ ;
lab = Stl@StringForm[
  "a = ` ` f = ` `  $\theta_0 = \pi/64$  \nHigh enough driver frequency and the \npendulum
  remains inverted and oscillates about vertical", a, f,  $\theta_0$ ];
Plot[ $\theta_{\text{Exact}}[\tau]$ , { $\tau$ , 0,  $\tau_{\text{Max}}$ }, PlotStyle -> BLACK,
  PlotLabel -> Stl[lab], AxesLabel -> {Stl[" $\tau$ "], " $\theta$ "},
  PlotRange -> {{0,  $\tau_{\text{Max}}$ }, {- $\pi/16$ ,  $\pi/16$ }},
  Ticks -> {Automatic, {- $\pi/16$ , - $\pi/32$ , 0,  $\pi/32$ ,  $\pi/16$ }}]]
```

$$a = 0.1 \quad f = 20 \quad \theta_0 = \frac{\pi}{64}$$

High enough driver frequency and the
pendulum remains inverted and oscillates about vertical



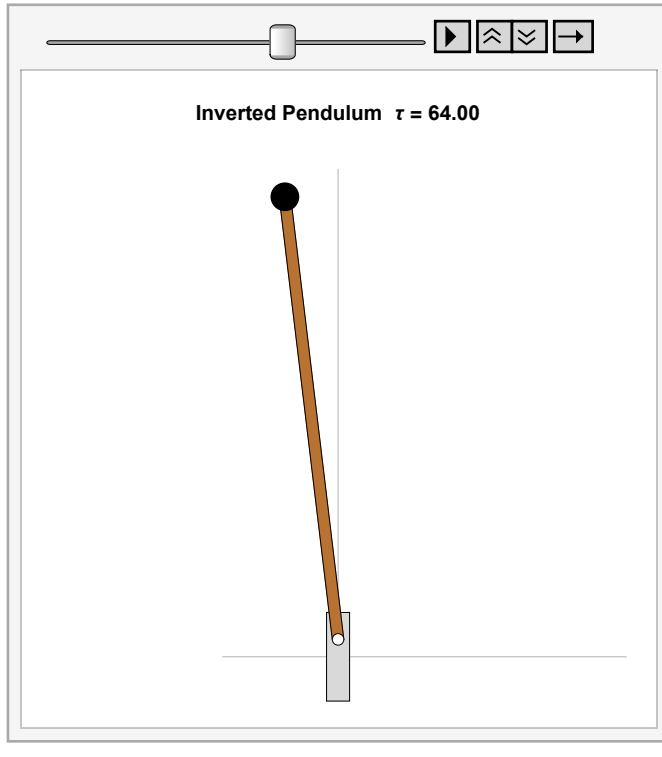
Out[32]=

An animation

In[36]=

```
Module[{τ = 0, A0 = 0.1, L = 1.0, θ0 = π/64, f = 20, a, θ, δz, frames},
  a = A0/L;
  frames = Table[δz = A0 Cos[τ];
    θ = 
$$\frac{\theta_0 \text{MathieuC}\left[-\frac{4}{f^2}, -2a, \frac{\tau}{2}\right]}{\text{MathieuC}\left[-\frac{4}{f^2}, -2a, \theta\right]}$$
;
    ShowPendulum[{τ, θ, δz}, A0, L], {τ, 0, 100, 2}];
  ListAnimate[frames]]
```

Out[36]=



3 Analysis of characteristic features of the motion

The following perturbation approach explains the qualitative features of the inverted pendulum motion. A quantitatively accurate description requires a more sophisticated treatment of the Mathieu equation that lies beyond the scope of my current interest. Pointers to the literature are provided at the end of this section.

In[37]=

```
w3[1] = MapEqn[ (# f^2) &, w1[15]] // Expand
```

Out[37]=

```
f^2 θ''[τ] == θ[τ] - a f^2 Cos[τ] θ[τ]
```

As stated before, the parameters satisfy $a \ll 1$, $f \gg 1$. But the inverted state will also require $(a f^2) \gg 1$. This can be satisfied by the ordering $f \sim 1/\epsilon$, $a \sim \epsilon$ with $\epsilon \ll 1$

In[38]:= `w3[2] = w3[1] /. f -> f/ε /. a -> a ε`

Out[38]=
$$\frac{f^2 \theta''[\tau]}{\epsilon^2} == \theta[\tau] - \frac{a f^2 \cos[\tau] \theta[\tau]}{\epsilon}$$

The visualization of the inverted pendulum in the previous section implies that the lowest order θ motion is approximately steady with a small amount of modulation at the driver frequency. This suggests the ansatz

In[39]:= `w3[3] = w3[2] /. θ -> ((C[#] + ε b Cos[#]) &) // ExpandAll`

Out[39]=
$$-\frac{b f^2 \cos[\tau]}{\epsilon} + \frac{f^2 C''[\tau]}{\epsilon^2} == b \epsilon \cos[\tau] - a b f^2 \cos[\tau]^2 + C[\tau] - \frac{a f^2 \cos[\tau] C[\tau]}{\epsilon}$$

The lowest order equations in ϵ are

In[40]:= `w3[4] = {Coefficient[#, ε^-2] & /@ w3[3], Coefficient[#, ε^-1] & /@ w3[3]};
w3[4] // ColumnForm`

Out[41]=
$$f^2 C''[\tau] == 0$$

$$-b f^2 \cos[\tau] == -a f^2 \cos[\tau] C[\tau]$$

The lowest order equation implies C is constant

In[42]:= `w3[5] = w3[4][[1]]`

Out[42]=
$$f^2 C''[\tau] == 0$$

The second order equation provides a relationship between b and $C(\tau)$

In[43]:= `w3[6] = w3[4][[2]]`

Out[43]=
$$-b f^2 \cos[\tau] == -a f^2 \cos[\tau] C[\tau]$$

In[44]:= `w3[7] = Sol[w3[6], C[τ]] // RE`

Out[44]=
$$C[\tau] == \frac{b}{a}$$

Thus, the ansatz is refined to

In[46]:= `θ[τ] -> (C[τ] + ε b Cos[τ]) /. Sol[w3[7], b] // Simplify`

Out[46]=
$$\theta[\tau] \rightarrow (1 + a \epsilon \cos[\tau]) C[\tau]$$

The calculation proceeds by assuming that C varies on a slow time scale with respect to the time scale of the driver motion.

Reconsider the original equation

In[47]:=

w1[15]

Out[47]:=

$$\theta''[\tau] == \frac{\theta[\tau]}{f^2} - a \text{Cos}[\tau] \theta[\tau]$$

I am interested in the slow — or average — motion of the pendulum. The procedure for calculating this average motion is more clear if each side of this equation is considered separately.

Consider the lhs and introduce the ansatz.

In[48]:=

w3[8] = w1[15][[1]] == (w1[15][[1]] /. $\theta \rightarrow ((C[\#] (1 + a \text{Cos}[\#])) \&)$ // ExpandAll)

Out[48]:=

$$\theta''[\tau] == -a \text{Cos}[\tau] C[\tau] - 2 a \text{Sin}[\tau] C'[\tau] + C''[\tau] + a \text{Cos}[\tau] C''[\tau]$$

Assume that C varies on a much slower time scale than τ . This can be operationally accomplished by

In[49]:=

w3[9] = w3[8] /. $C \rightarrow ((C_\theta[\epsilon \#]) \&)$ /. $\epsilon \tau \rightarrow \tau_{\text{slow}}$ /. $\epsilon \rightarrow 1$ // Expand

Out[49]:=

$$\theta''[\tau] == -a \text{Cos}[\tau] C_\theta[\tau_{\text{slow}}] - 2 a \text{Sin}[\tau] C_\theta'[\tau_{\text{slow}}] + C_\theta''[\tau_{\text{slow}}] + a \text{Cos}[\tau] C_\theta''[\tau_{\text{slow}}]$$

Average over the τ time scale to find the average of $\theta''[\tau]$

In[50]:=

w3[10] = MapEqn[$\frac{1}{2\pi}$ Integrate[#, { τ , 0, 2 π }] &, w3[9]]

Out[50]:=

$$\frac{\int_0^{2\pi} \theta''[\tau] d\tau}{2\pi} == C_\theta''[\tau_{\text{slow}}]$$

Perform these same operations on the rhs

In[51]:=

**w3[12] =
w3[1][[2]] /. $\theta \rightarrow ((C[\#] (1 + a \text{Cos}[\#])) \&)$ /. $C \rightarrow ((C_\theta[\epsilon \#]) \&)$ /. $\epsilon \tau \rightarrow \tau_{\text{slow}}$ /.
 $\epsilon \rightarrow 1$ // Expand // Map[$\frac{1}{2\pi}$ Integrate[#, { τ , 0, 2 π }] &, #] &**

Out[51]:=

$$C_\theta[\tau_{\text{slow}}] - \frac{1}{2} a^2 f^2 C_\theta[\tau_{\text{slow}}]$$

Recombine the lhs and rhs

In[52]:=

w3[13] = w3[10][[2]] == w3[12]

Out[52]:=

$$C_\theta''[\tau_{\text{slow}}] == C_\theta[\tau_{\text{slow}}] - \frac{1}{2} a^2 f^2 C_\theta[\tau_{\text{slow}}]$$

Define a frequency of convenience

In[53]= `def[Ω] = Ω² == -Coefficient[w3[13][2], Cθ[τslow]]`

Out[53]=
$$\Omega^2 == -1 + \frac{a^2 f^2}{2}$$

The equation for the slow time scale motion is

In[54]= `w3[14] = w3[13] /. Solve[def[Ω], a][[1]] // Expand`

Out[54]=
$$C\theta''[\tau_{slow}] == -\Omega^2 C\theta[\tau_{slow}]$$

I solve this with nominal initial conditions

In[55]= `w3[15] == DSolve[{w3[14], Cθ'[0] == 0, Cθ[0] == α}, Cθ[τslow], τslow][[1, 1]] // RE`

Out[55]=
$$w3[15] == (C\theta[\tau_{slow}] == \alpha \cos[\tau_{slow} \Omega])$$

Thus, the slow time scale or average motion is oscillatory about vertical as seen in the diagram of Section 2.

As Morin points out, a more detailed analysis of this problem may be found in *On the Dynamic Stabilization of the of an Inverted Pendulum*, E. Butikov, Am. J. Phys. 69 (6), June, 2001 [I found this on the web]. Butikov details the physical mechanism by which the pendulum remains inverted, discussing the average torque imposed on the pendulum caused by the driver.

A quantitatively accurate perturbation theory involves a nontrivial analysis of Mathieu's equation, *Small Oscillations of a Stabilized Inverted Pendulum*, D. J. Ness, Am. J. Phys. 35, 964,(1967) [also on web] and references therein.

Graphics

ShowPendulum is used to animate the inverted pendulum

In[34]:=

```

Clear[ShowPendulum];
ShowPendulum[{\tau_, \theta_, \delta z_}, A\theta_, L_] :=
Module[{\delta x, \phi, axes, bracket, connector, pendulum, copperColor, lab},
  copperColor = RGBColor[{184, 115, 51}/255];
  \phi = -\theta;
  \delta x = 0.1 A\theta;
  axes =
    {GrayLevel[0.7], Line[{{-0.1 L, 0}, {0.25 L, 0}}, Line[{{0, 0}, {0, 1.1 L}}]};
  connector = OC[{0, \delta z}, Black];
  (* structure representing the driver of vertical motion *)
  bracket =
    {EdgeForm[{Thin, Black}], {LightGray, Rectangle[{-\delta x, -A\theta}, {\delta x, A\theta}]}];
  (* upright pendulum *)
  pendulum =
    {{EdgeForm[{Thin, Black}], {copperColor, Rectangle[{-\delta x/2, 0}, {\delta x/2, L}]}},
    {Black, PointSize[0.05], Point[{0, L}]}];
  (* rotated and translated pendulum *)
  pendulum = Rotate[pendulum, \phi, {0, 0}];
  pendulum = Translate[pendulum, {0, \delta z}];

  lab = Stl@StringForm["Inverted Pendulum \tau = ``", NF2[\tau]];
  Graphics[{axes, bracket, pendulum, connector},
    AspectRatio \to 1.0, PlotLabel \to lab, ImageSize \to 300,
    PlotRange \to {{-0.25 L, 0.25 L}, {-0.1 L, 1.2 L}}]

```

Generation of the initial diagram

In[56]=

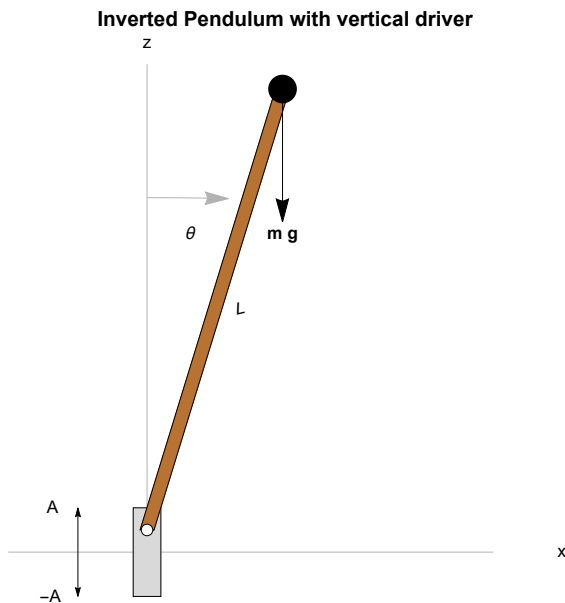
```

Module[{O, P, A, L,  $\theta$ ,  $\delta x$ ,  $\delta z$ , horizontal, vertical, bracket, connector,
  pendulum, arc, arcAngle, lenVec, forceVector, doubleArrow, copperColor, lab},
  copperColor = RGBColor[{184, 115, 51}/255];
  {A, L,  $\theta$ , lenVec} = {0.1, 1, 15  $\pi$ /32, 0.3};
  { $\delta x$ ,  $\delta z$ } = {0.01, A/2};
  {O, P} = {{0, 0}, {0,  $\delta z$ } + L {Cos[ $\theta$ ], Sin[ $\theta$ ]}};
  arc = {GrayLevel[0.7], QArcArrow2DLabeled[0.8 L,  $\pi$ /2,  $\theta$  +  $\pi$ /256, " $\theta$ "]};
  horizontal =
    {GrayLevel[0.7], Line[{{-0.1 L, 0}, {0.25 L, 0}}], {Black, Text["x", {0.3 L, 0}]}];
  vertical = {GrayLevel[0.7], Line[{0, 0, 1.1 L}], {Black, Text["z", {0, 1.15 L}]}];
  connector = OC[{0,  $\delta z$ }, Black];
  (* structure representing the driver of vertical motion *)
  bracket = {EdgeForm[{Thin, Black}], {LightGray, Rectangle[{- $\delta x$ , -A}, { $\delta x$ , A}]}];
  (* upright pendulum *)
  pendulum =
    {{EdgeForm[{Thin, Black}], {copperColor, Rectangle[{- $\delta x$ /2, 0}, { $\delta x$ /2, L}]}},
    {Black, PointSize[0.05], Point[{0, L}], Text["L", {2  $\delta x$ , L/2}]}];
  (* rotated and translated pendulum *)
  pendulum = Rotate[pendulum, -( $\pi$ /2 -  $\theta$ ), {0, 0}];
  pendulum = Translate[pendulum, {0,  $\delta z$ };
  forceVector = LabeledVector[{P, P + {0, -lenVec}}, "m g", 1.1];
  doubleArrow =
    {Black, Arrowheads[{-0.015, 0.015}], Arrow[{{-0.05, -A}, {-0.05, A}],
    Text["-A", {-0.07, -A}], Text["A", {-0.07, A}]}];

  lab = St1["Inverted Pendulum with vertical driver"];
  Graphics[{horizontal, vertical, bracket, pendulum, connector, arc, forceVector,
    doubleArrow}, AspectRatio  $\rightarrow$  1.0, PlotLabel  $\rightarrow$  lab, ImageSize  $\rightarrow$  300]

```

Out[56]=



In[3]:=

```

Clear[QArcArrow2DLabeled];
QArcArrow2DLabeled[r_,  $\theta_S$ _,  $\theta_F$ _, text_] :=
  {Arrow@Table[CoordinateTransform["Polar"  $\rightarrow$  "Cartesian", {r,  $\xi$ }],
    { $\xi$ ,  $\theta_S$ ,  $\theta_F$ , Sign[ $\theta_F - \theta_S$ ]  $\frac{\pi}{128}$ }},
  {Black, Text[text, CoordinateTransform["Polar"  $\rightarrow$  "Cartesian",
    {0.9 r,  $\frac{1}{2} (\theta_S + \theta_F)$ }]}}}

```

In[5]:=

```

Clear[LabeledVector];
LabeledVector[{tail_, head_}, lab_, frac_] :=
  Module[{labPosn},
    labPosn = tail + frac (head - tail);
    {Arrow[{tail, head}], Text[Stl[lab], labPosn]} ]

```