

Hanging Chain 10-01-16

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Initialization: Be sure the files *NTGStylesheet2.nb* and *NTGUtilityFunctions.m* is are in the same directory as that from which this notebook was loaded. Then execute the cell immediately below by mousing left on the cell bar to the right of that cell and then typing “shift” + “enter”. Respond “Yes” in response to the query to evaluate initialization cells.

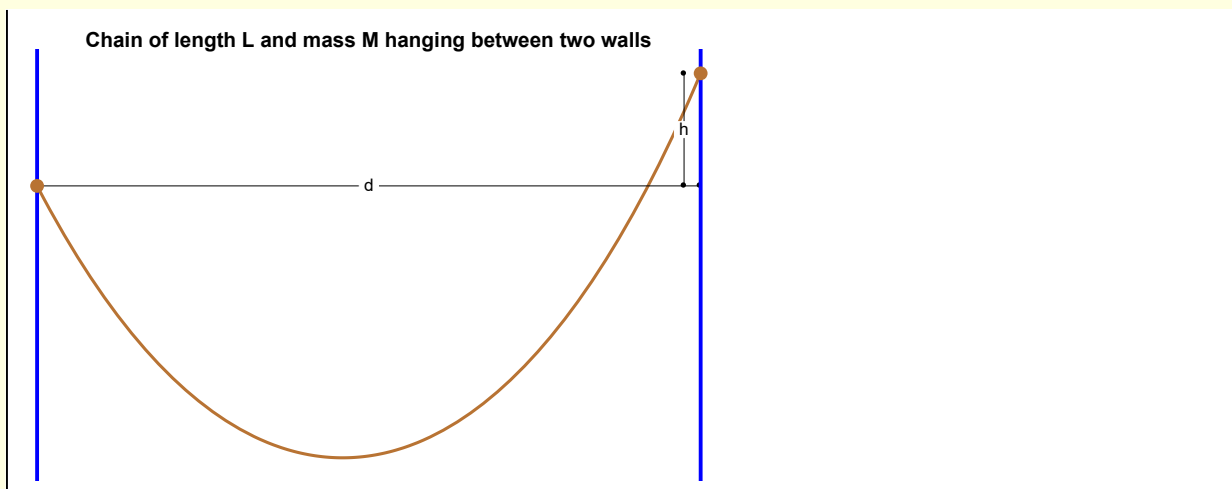
In[29]:=

```
SetDirectory[NotebookDirectory[]];
(* set directory where source files are located *)
SetOptions[EvaluationNotebook[], (* load the StyleSheet *)
  StyleDefinitions -> Get["NTGStylesheet2.nb"]];
Get["NTGUtilityFunctions.m"]; (* Load utilities package *)
```

Purpose

I use Mathematica to work through the details of solving the classic problem of determining the shape of a hanging chain. The Wikipedia entry <https://en.wikipedia.org/wiki/Catenary> has some fascinating history that even involves Thomas Jefferson!

I probably first encountered this problem as an undergraduate. But that is lost in the mists of time and I re-encountered it while working through problems in *Morin's Introduction to Classical Mechanics* -- problem 2.8. It is also problem number 75 of the Harvard undergraduate problem series.

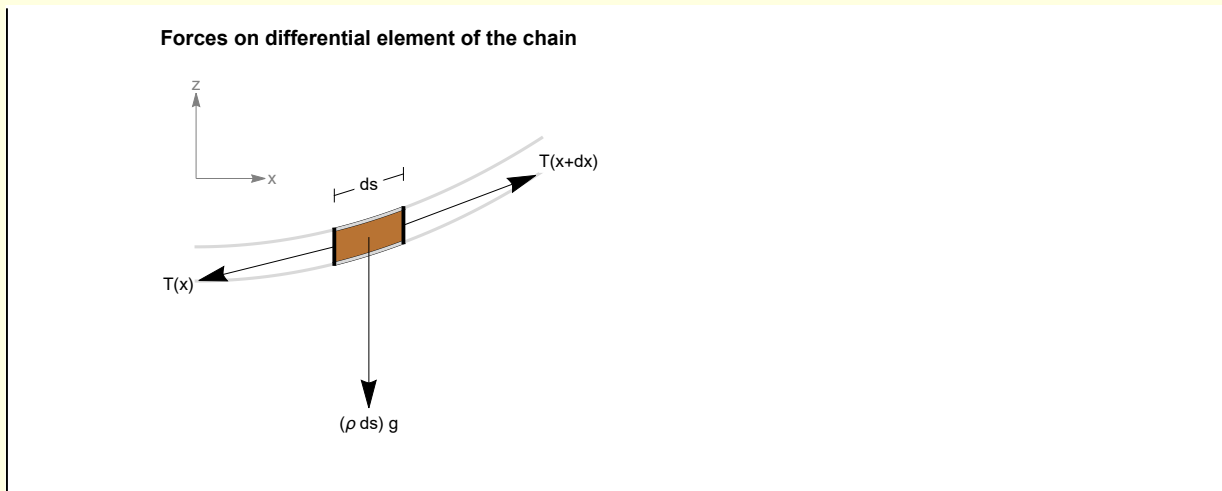


I Derivation of shape equation from statics

The differential equation describing the shape of the chain is

$$z''(x) = \kappa \sqrt{z'(x)^2 + 1}$$

I follow Morin and derive this differential equation from statics. It can also be derived from the calculus of variations, although the required manipulations are less straightforward. Also, there is an interesting discrete approach to this problem illustrated as an example of the capabilities of Mathematica 11.0 <http://www.wolfram.com/language/11/symbolic-and-numeric-calculus/model-a-hanging-chain.html?product=mathematica>. Discrete methods, in particular by Huygens, were important in the historical development of this topic — <http://jcsites.juniata.edu/faculty/bukowski/leiden/cmj002-011.pdf>



A key observation is that the horizontal tension in a hanging chain is constant (call it \mathcal{T}) that will have to be determined. Thus for the differential element of the chain illustrated

$$T_x(x) = \mathcal{T}$$

Vertical force balance on a differential element of the chain requires

$$dT_z(x) = (\rho ds) g$$

where $\rho = M/L$, with M and L the mass and length of the chain.

Mathematica representation

In[32]=

$$\mathbf{w1}[1] = dT_z == \rho ds [x] g$$

Out[32]=

$$dT_z == g \rho ds [x]$$

The differential length element of the hanging chain described by $z = z(x)$ is

$$ds = \sqrt{dx^2 + dz^2} = dx \sqrt{1 + \left(\frac{dz}{dx}\right)^2}$$

In[33]=

$$\mathbf{w1}[2] = \mathbf{ds}[x] == \mathbf{dx} \sqrt{1 + \mathbf{D}[z[x], x]^2}$$

Out[33]=

$$\mathbf{ds}[x] == \mathbf{dx} \sqrt{1 + z'[x]^2}$$

Further, the tension must be aligned along the differential element (else it would rotate)

$$\frac{T_z(x)}{T_x(x)} = \frac{T_z(x)}{\mathcal{T}} = \frac{dz}{dx}$$

In[34]=

$$\mathbf{w1}[3] = T_z[x] == \mathcal{T} \mathbf{D}[z[x], x]$$

Out[34]=

$$T_z[x] == \mathcal{T} z'[x]$$

The length of the chain is

$$L = \int_0^d dx ds(x) = \int_0^d dx \sqrt{1 + z'[x]^2}$$

In[35]=

$$\mathbf{w1}[4] = L == \mathbf{Integrate}[\sqrt{1 + \mathbf{D}[z[x], x]^2}, \{x, 0, d\}]$$

Out[35]=

$$L == \int_0^d \sqrt{1 + z'[x]^2} dx$$

A second condition is that the vertical distance between the attachment points is specified

$$z(d) - z(0) = \lambda$$

In[36]=

$$\mathbf{w1}[5] = z[d] - z[0] == h$$

Out[36]=

$$-z[0] + z[d] == h$$

The objective is to derive a differential equation for $z(x)$. I manipulate the above equations

In[37]=

$$\mathbf{w1}[6] = \mathbf{w1}[1] /. \mathbf{dT}_z \rightarrow \mathbf{D}[T_z[x], x] \mathbf{dx}$$

Out[37]=

$$\mathbf{dx} T_z'[x] == g \rho \mathbf{ds}[x]$$

Then

In[38]=

$$\mathbf{w1}[7] = \mathbf{w1}[6] /. \mathbf{Sol}[\mathbf{w1}[2], \mathbf{ds}[x]] // \mathbf{RE}$$

Out[38]=

$$\mathbf{dx} T_z'[x] == \mathbf{dx} g \rho \sqrt{1 + z'[x]^2}$$

Notice also

In[39]:= **dTzdxRule = MapEqn[D[#, x] &, w1[3]] // ER**

Out[39]= $T_z'[x] \rightarrow \mathcal{T} z''[x]$

So

In[40]:= **w1[8] = w1[7] /. dTzdxRule**

Out[40]= $dx \mathcal{T} z''[x] == dx g \rho \sqrt{1 + z'[x]^2}$

In[41]:= **w1[9] = MapEqn[(# / (\mathcal{T} dx)) &, w1[8]] /. \rho \rightarrow M / L**

Out[41]= $z''[x] == \frac{g M \sqrt{1 + z'[x]^2}}{L \mathcal{T}}$

Introduce a single parameter κ

In[42]:= **def[\kappa] = \kappa == \frac{g M}{L \mathcal{T}};**

and the desired differential equation is obtained.

In[43]:= **w1[10] = w1[9] /. Sol[def[\kappa], g]**

Out[43]= $z''[x] == \kappa \sqrt{1 + z'[x]^2}$

2 Solution

The solution of the basic differential equation

$$z''(x) = \kappa \sqrt{z'(x)^2 + 1}$$

is trivial. However application of the boundary and constraint conditions leads to equations that require numerical solution.

In[44]:= **w2[1] = DSolve[w1[10], z[x], x] [[1, 1]] /. C[1] \rightarrow c1 /. C[2] \rightarrow c2 // Simplify**

Out[44]= $z[x] \rightarrow c2 + \frac{\text{Cosh}[c1 + x \kappa]}{\kappa}$

In[45]:= **w2[2] = {w2[1] /. x \rightarrow 0 /. z[0] \rightarrow 0, w2[1] /. x \rightarrow d /. z[d] \rightarrow h} // RE**

Out[45]= $\left\{ 0 == c2 + \frac{\text{Cosh}[c1]}{\kappa}, h == c2 + \frac{\text{Cosh}[c1 + d \kappa]}{\kappa} \right\}$

In[46]:= **w2[3] = Solve[w2[2][[1]], c2][[1, 1]]**

Out[46]=
$$c2 \rightarrow -\frac{\text{Cosh}[c1]}{\kappa}$$

In[47]:= **w2[4] = w2[1] /. w2[3] // RE**

Out[47]=
$$z[x] == -\frac{\text{Cosh}[c1]}{\kappa} + \frac{\text{Cosh}[c1 + x \kappa]}{\kappa}$$

It remains to determine the remaining constants $c1$ and κ . One condition obtains fixed length of the chain, a second condition comes the specified vertical separation of the endpoints.

2A Condition from length of chain

The solution must satisfy

$$L = \int_0^d dx ds(x) = \int_0^d dx \sqrt{1 + z'[x]^2}$$

In[48]:= **w2A[1] = w1[4]**

Out[48]=
$$L == \int_0^d \sqrt{1 + z'[x]^2} dx$$

From the solution

In[49]:= **dzdxRule = MapEqn[D[#, x] &, w2[4]] // ER**

Out[49]=
$$z'[x] \rightarrow \text{Sinh}[c1 + x \kappa]$$

In[50]:= **w2A[2] = $\sqrt{1 + z'[x]^2}$ == ($\sqrt{1 + z'[x]^2}$ /. dzdxRule // Simplify // PowerExpand)**

Out[50]=
$$\sqrt{1 + z'[x]^2} == \text{Cosh}[c1 + x \kappa]$$

Then

In[51]:= **w2A[3] = w2A[1] /. (w2A[2] // ER)**

Out[51]=
$$L == \frac{-\text{Sinh}[c1] + \text{Sinh}[c1 + d \kappa]}{\kappa}$$

In[52]:= **w2A[4] = MapEqn[(Simplify[# \kappa]) &, w2A[3]] // Reverse**

Out[52]=
$$-\text{Sinh}[c1] + \text{Sinh}[c1 + d \kappa] == L \kappa$$

2B Condition from different connection points

The connection points are separated by h

$$z(d) - z(0) = h$$

Recall that the solution is

$$\text{In[53]:= } w2B[1] = w1[5]$$

$$\text{Out[53]:= } -z[0] + z[d] == h$$

$$\text{In[54]:= } w2B[2] = w2B[1] /. z[x_] \to w2[4][[2]]$$

$$\text{Out[54]:= } -\frac{\text{Cosh}[c1]}{\kappa} + \frac{\text{Cosh}[c1 + d \kappa]}{\kappa} == h$$

$$\text{In[55]:= } w2B[3] = \text{MapEqn}[(\text{Simplify}[\# \kappa]) \&, w2B[2]]$$

$$\text{Out[55]:= } -\text{Cosh}[c1] + \text{Cosh}[c1 + d \kappa] == h \kappa$$

3 Solving the constraint equations

$$\text{In[56]:= } w3[1] = \{w2A[4], w2B[3]\}$$

$$\text{Out[56]:= } \{-\text{Sinh}[c1] + \text{Sinh}[c1 + d \kappa] == L \kappa, -\text{Cosh}[c1] + \text{Cosh}[c1 + d \kappa] == h \kappa\}$$

Later, I'll show these two equations can be reduced to a single equation. For the time being I proceed with a numerical solution of the two equations as they stand.

I start by considering the size of some of the terms.

Note

$$\text{In[58]:= } w3[2] = L \kappa /. \text{Sol}[\text{def}[\kappa], \kappa]$$

$$\text{Out[58]:= } \frac{g M}{\tau}$$

It is reasonable to expect the ratio of the weight of the chain to be greater than the horizontal tension. I introduce a parameter f

$$\text{In[59]:= } \text{def}[f] = f == \frac{g M}{\tau}$$

$$\text{Out[59]:= } f == \frac{g M}{\tau}$$

where f is expected to be > 1

In[61]=

w3[3] = w3[1] /. Sol[def[x], x] /. Sol[def[f], g]

Out[61]=

$$\left\{ -\text{Sinh}[c1] + \text{Sinh}\left[c1 + \frac{df}{L}\right] == f, -\text{Cosh}[c1] + \text{Cosh}\left[c1 + \frac{df}{L}\right] == \frac{fh}{L} \right\}$$

Next I introduce some length scales by choosing the horizontal separation of the walls as a reference length

In[62]=

w3[4] = w3[3] /. h → hd /. L → ld

Out[62]=

$$\left\{ -\text{Sinh}[c1] + \text{Sinh}\left[c1 + \frac{f}{l}\right] == f, -\text{Cosh}[c1] + \text{Cosh}\left[c1 + \frac{f}{l}\right] == \frac{fh}{l} \right\}$$

where the ratio $l = L/d > 1$ and the ratio $h = h/d < 1$.

In[63]=

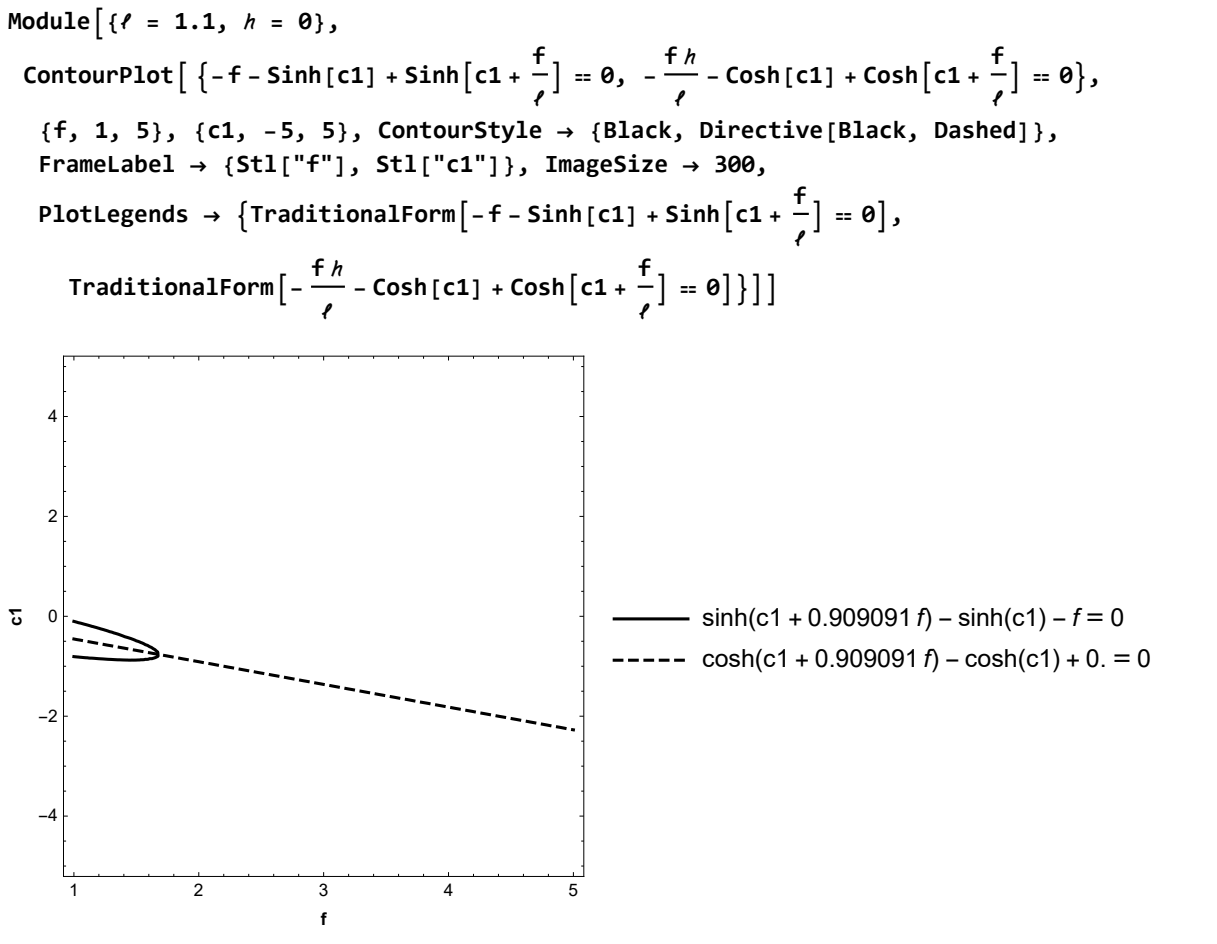
w3[5] = StandardizeEqn /@ w3[4]

Out[63]=

$$\left\{ -f - \text{Sinh}[c1] + \text{Sinh}\left[c1 + \frac{f}{l}\right] == 0, -\frac{fh}{l} - \text{Cosh}[c1] + \text{Cosh}\left[c1 + \frac{f}{l}\right] == 0 \right\}$$

Given some general feel for the nominal values of parameters, I can use ContourPlot to locate a possible root of the coupled equations. I start with the case when there is no vertical separation of the connection points.

In[69]=



Out[69]=

and then FindRoot to determine the actual root.

In[70]=

```

Module[{l = 1.1, h = 0, fGuess = 1.5, c1Guess = -0.5, root},
  root = FindRoot[{-f - Sinh[c1] + Sinh[c1 + f/l] == 0, -f/h - Cosh[c1] + Cosh[c1 + f/l] == 0},
    {f, fGuess}, {c1, c1Guess}]

```

Out[70]=

{f -> 1.67948, c1 -> -0.763401}

I develop a dimensionless form for z(x) in order to graphically represent the solution.

In[72]=

```

w3[6] = z[X] == w2[4][[2]] /. Sol[def[x], κ] /. Sol[def[f], g] /. x -> X d /. L -> l d

```

$$z[X] == -\frac{d \, l \, \text{Cosh}[c1]}{f} + \frac{d \, l \, \text{Cosh}\left[c1 + \frac{f X}{l}\right]}{f}$$

Out[72]=


```
In[73]:= w3[7] = w3[6] /. z -> ((Z[#] d) &);
w3[7] = MapEqn[(Simplify[#/d] &), w3[7]]
```

$$\text{Out[74]= } Z[X] = \frac{1}{f} \left(-\text{Cosh}[c1] + \text{Cosh}\left[c1 + \frac{fX}{l}\right] \right)$$

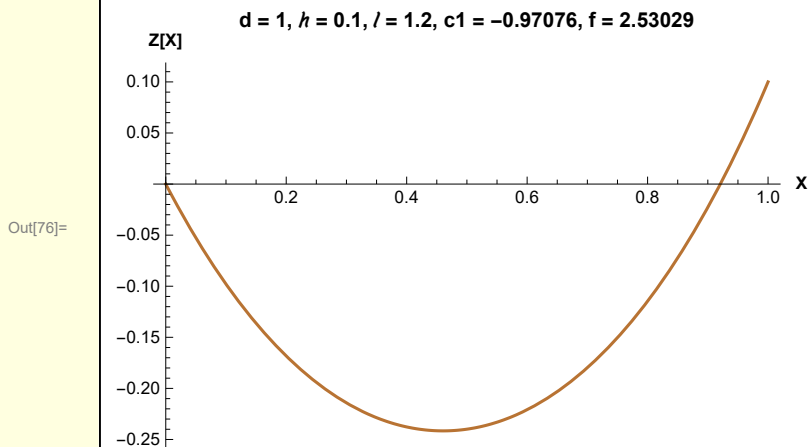
I also express the length of the chain in terms of these variables.

```
In[75]:= w3[8] =
L == (w3[7][[2]] /. Sol[def[x], x] /. Sol[def[f], g] /. x -> X d /. L -> l d) /. d -> 1
```

$$\text{Out[75]= } L = \frac{1}{f} l \left(-\text{Cosh}[c1] + \text{Cosh}\left[c1 + \frac{fX}{l}\right] \right)$$

It's straightforward to extend to $h > 0$ although better initial conditions may be required for RootFinder to converge.

```
In[76]:= Module[{l = 1.2, h = 0.1, fGuess = 1.5, c1Guess = -0.5,
  copperColor = RGBColor[{184, 115, 51}/255], root, c1, f, L, cc1, ff, ZFcn, lab},
  root = FindRoot[{-f - Sinh[c1] + Sinh[c1 + f/l] == 0, -f/h - Cosh[c1] + Cosh[c1 + f/l] == 0},
    {f, fGuess}, {c1, c1Guess}];
  ZFcn[X_] := 1/f (-Cosh[c1] + Cosh[c1 + fX/l]) /. root;
  L = 1/f (-Sinh[c1] + Sinh[c1 + f/l]) /. root;
  {cc1, ff} = {c1, f} /. root;
  lab = Stl@StringForm["d = 1, h = ``, l = ``, c1 = ``, f = ``, h, l, cc1, ff];
  Plot[ZFcn[X], {X, 0, 1}, AxesLabel -> {Stl["X"], Stl["Z[X]"]},
    PlotLabel -> lab, PlotStyle -> copperColor]
```



4 Reduction to a single constraint equation

Another approach is to reduce the two condition equations to a single equation (one, however, that still requires numerical solution.)

```
In[77]:= w4[1] = {-Sinh[c1] + Sinh[c1 + d κ] == L κ, -Cosh[c1] + Cosh[c1 + d κ] == h κ}
```

```
Out[77]:= {-Sinh[c1] + Sinh[c1 + d κ] == L κ, -Cosh[c1] + Cosh[c1 + d κ] == h κ}
```

```
In[78]:= w4[2] = {MapEqn[(ExpandAll[#^2]) &, w4[1][[1]]], MapEqn[(ExpandAll[#^2]) &, w4[1][[2]]]}
```

```
Out[78]:= {Sinh[c1]^2 - 2 Sinh[c1] Sinh[c1 + d κ] + Sinh[c1 + d κ]^2 == L^2 κ^2,
Cosh[c1]^2 - 2 Cosh[c1] Cosh[c1 + d κ] + Cosh[c1 + d κ]^2 == h^2 κ^2}
```

```
In[79]:= w4[3] = w4[2] /. Cosh[x_]^2 -> 1 + Sinh[x]^2
```

```
Out[79]:= {Sinh[c1]^2 - 2 Sinh[c1] Sinh[c1 + d κ] + Sinh[c1 + d κ]^2 == L^2 κ^2,
2 - 2 Cosh[c1] Cosh[c1 + d κ] + Sinh[c1]^2 + Sinh[c1 + d κ]^2 == h^2 κ^2}
```

```
In[80]:= w4[4] = w4[3][[1, 1]] - w4[3][[2, 1]] == w4[3][[1, 2]] - w4[3][[2, 2]]
```

```
Out[80]:= -2 + 2 Cosh[c1] Cosh[c1 + d κ] - 2 Sinh[c1] Sinh[c1 + d κ] == -h^2 κ^2 + L^2 κ^2
```

```
In[81]:= w4[5] = MapEqn[Simplify, w4[4]]
```

```
Out[81]:= 2 (-1 + Cosh[d κ]) == (-h^2 + L^2) κ^2
```

```
In[84]:= w4[6] =
w4[5] /. Sol[def[κ], κ] /. ρ -> M/L /. Sol[def[f], g] /. L -> ℓ d /. h -> h d;
w4[6] = MapEqn[Simplify, w4[6]]
```

```
Out[85]:= 2 (-1 + Cosh[f/ℓ]) == f^2 (1 - h^2/ℓ^2)
```

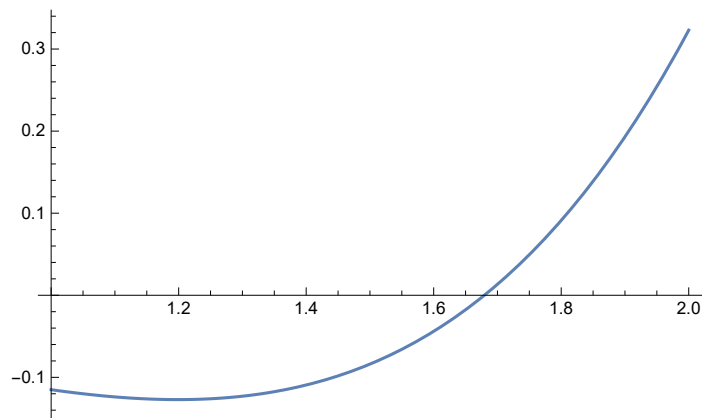
```
In[96]:= Clear[Fcn];
```

```
Fcn[f_, ℓ_, h_] := 2 (-1 + Cosh[f/ℓ]) - f^2 (1 - h^2/ℓ^2)
```

In[100]:=

```
Module[{l = 1.1, h = 0, fGuess = 1.5, root},
  Plot[Fcn[f, l, h], {f, 1, 2}, Axes → {Stl["f"], Stl["Fcn"]}]]
```

Out[100]=



In[101]:=

```
Module[{l = 1.1, h = 0, fGuess = 1.5, root},
  root = FindRoot[Fcn[f, l, h] == 0, {f, fGuess}][[1]]
```

Out[101]=

```
f → 1.67948
```

which agrees with the root found in section 3 for the same parameters. A practical advantage of this approach is that the root finder is less sensitive to the initial guess.

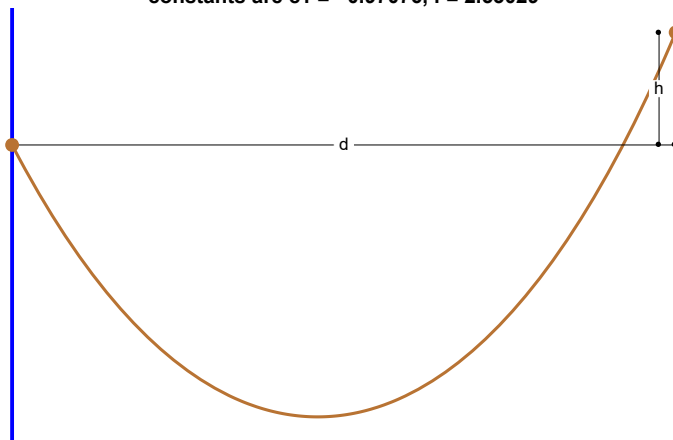
Visualizations

In[107]:=

```
Module[{l = 1.2, h = 0.1, fGuess = 1.5, c1Guess = -0.5,
  copperColor = RGBColor[{184, 115, 51}/255], root, c1, f,
  L, cc1, ff, ZFcn, lab, dLine, hLine, walls, connections},
  root = FindRoot[{-f - Sinh[c1] + Sinh[c1 +  $\frac{f}{l}$ ] == 0, - $\frac{f h}{l}$  - Cosh[c1] + Cosh[c1 +  $\frac{f}{l}$ ] == 0},
    {f, fGuess}, {c1, c1Guess}];
  ZFcn[X_] :=  $\frac{1}{f}$  (-Cosh[c1] + Cosh[c1 +  $\frac{f X}{l}$ ]) /. root;
  L =  $\frac{1}{f}$  (-Sinh[c1] + Sinh[c1 +  $\frac{f}{l}$ ]) /. root;
  {cc1, ff} = {c1, f} /. root;
  dLine = HDimensionLine[{0, 0}, {1, 0}, 0.0, " d ", {0, 0}];
  hLine = VDimensionLine[{0.975, 0}, {0.975, h}, 0.0, " h ", {0, 0}];
  walls = {BLUE, Line[{{0, -0.5}, {0, 0.5}}], Line[{{1, -0.5}, {1, 0.5}}]};
  connections = {copperColor, PointSize[0.02], Point[{0, 0}], Point[{1, h}]};
  lab = Stl@StringForm[
    "Chain of length L and mass M hanging between two walls\n
    parameters are d = 1, h = ``, l = ``\n
    constants are c1 = ``, f = ``", h, l, cc1, ff];
  Plot[ZFcn[X], {X, 0, 1}, Axes → False, AxesLabel → {Stl["X"], Stl["Z[X]"]},
    PlotStyle → copperColor,
    Epilog → {dLine, hLine, walls, connections}, PlotLabel → lab]
```

Chain of length L and mass M hanging between two walls
 parameters are d = 1, h = 0.1, l = 1.2
 constants are c1 = -0.97076, f = 2.53029

Out[107]=



In[113]:=

```

Module[{ $\delta = 0.02$ ,  $x_C = 0.25$ , copperColor = RGBColor[{184, 115, 51}/255],  $x_L$ ,  $x_R$ ,  $\alpha$ ,
   $B$ ,  $C$ , endCaps, gVec, TVecL, TVecR, dimLine, axes, lab, Curve, Tangent, G},

  Curve[x_] := Cosh[x];
  Tangent[x_] := Sinh[x];

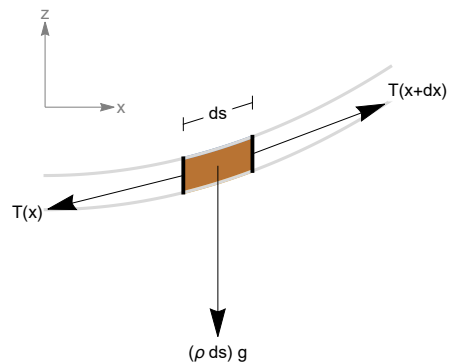
  axes =
    {Gray, With[{start = {0.0, 1.10}, mag = 0.1, angle = 0, text = "x", mult = 1.1},
      {Arrowheads[Small], QLabelledArrow[start, mag, angle, text, mult],
        QLabelledArrow[start, mag,  $\pi/2$ , "z", mult]}}];

  (* Background plot *)
  G[1] = Plot[{Curve[x] +  $\delta$ , Curve[x] -  $\delta$ }, {x, 0, 0.5}, PlotStyle  $\rightarrow$  LightGray,
    PlotRange  $\rightarrow$  {{ $x_C - 0.5$ ,  $x_C + 0.5$ }, {1 - 0.5, 1 + 0.5}}];

  (* Main plot *)
  { $x_L$ ,  $x_R$ } =  $x_C + \{-0.05, 0.05\}$ ;
  (* graphics items *)
  endCaps = {BLACK, Line[{{ $x_L$ , Curve[ $x_L$ ] +  $\delta$ }, { $x_L$ , Curve[ $x_L$ ] -  $\delta$ }}],
    Line[{{ $x_R$ , Curve[ $x_R$ ] +  $\delta$ }, { $x_R$ , Curve[ $x_R$ ] -  $\delta$ }}]};
  dimLine = HDimensionLine[{ $x_L$ , Curve[ $x_L$ ] + 3  $\delta$ },
    { $x_R$ , Curve[ $x_R$ ] + 3  $\delta$ }, 0.0075, " ds ", {0, 0}];
  gVec = With[{start = { $x_C$ , Curve[ $x_C$ ]}, mag = 0.2,
    angle =  $-\pi/2$ , text = " $(\rho ds) g$ ", mult = 1.1},
    QLabelledArrow[start, mag, angle, text, mult]];
  TVecL = With[{start = { $x_L$ , Curve[ $x_L$ ]}, mag = 0.2,
    angle = ArcTan@Tangent[ $x_L$ ] +  $\pi$ , text = "T(x)", mult = 1.15},
    QLabelledArrow[start, mag, angle, text, mult]];
  TVecR = With[{start = { $x_R$ , Curve[ $x_R$ ]}, mag = 0.2,
    angle = ArcTan@Tangent[ $x_R$ ], text = "T(x+dx)", mult = 1.25},
    QLabelledArrow[start, mag, angle, text, mult]];
  lab = Stl@StringForm["Forces on differential element of the chain"];
  G[2] = Plot[{Curve[x] +  $\delta$ , Curve[x] -  $\delta$ }, {x,  $x_L$ ,  $x_R$ },
    PlotRange  $\rightarrow$  {{ $x_C - 0.5$ ,  $x_C + 0.5$ }, {1 - 0.25, 1 + 0.25}},
    Filling  $\rightarrow$  {1  $\rightarrow$  {{2}, copperColor}}, Epilog  $\rightarrow$ 
    {axes, endCaps, gVec, TVecL, TVecR, dimLine}, Axes  $\rightarrow$  None, PlotLabel  $\rightarrow$  lab];
  Show[{G[2], G[1]}]

```

Forces on differential element of the chain



Out[113]=

In[109]=

```
Clear[QLabeledArrow];
QLabeledArrow[start_, mag_, angle_, text_, mult_] :=
  With[{end = start + mag {Cos[angle], Sin[angle]},
        textPos = start + mult mag {Cos[angle], Sin[angle]}},
    {Arrow[{start, end}], Text[text, textPos]}];
```

In[111]=

```
Clear[QLabeledArrow2];
QLabeledArrow2[start_, fin_, text_, mult_] :=
  {Arrow[{start, fin}], Text[text, start + mult (fin - start)]}
```

In[103]=

```
VDimensionLine[start_, fin_, endFrac_, text_, off_] :=
  {Directive[Black, Thin], Line[{start, fin}],
   Line[{start - {endFrac, 0}, start + {endFrac, 0}}],
   Line[{fin - {endFrac, 0}, fin + {endFrac, 0}}], Text[text,  $\frac{\text{start} + \text{fin}}{2} + \text{off}$ ]};

HDimensionLine[start_, fin_, endFrac_, text_, off_] :=
  {Directive[Black, Thin], Line[{start, fin}],
   Line[{start - {0, endFrac}, start + {0, endFrac}}],
   Line[{fin - {0, endFrac}, fin + {0, endFrac}}], Text[text,  $\frac{\text{start} + \text{fin}}{2} + \text{off}$ ]};
```