Black Scholes PDE Derivation cross currency 01-03-11

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Initialization: Be sure the files *NTGStylesheet2.nb* and *NTGUtilityFunctions.m* is are in the same directory as that from which this notebook was loaded. Then execute the cell immediately below by mousing left on the cell bar to the right of that cell and then typing “shift” + “enter”. Respond “Yes” in response to the query to evaluate initialization cells.

In[3]:=
```
SetDirectory[NotebookDirectory[]];
(* set directory where source files are located *)
SetOptions[EvaluationNotebook[], (* load the StyleSheet *)
 StyleDefinitions → Get["NTGStylesheet2.nb"]];
Get["NTGUtilityFunctions.m"]; (* Load utilities package *)
```

Derivation of the Black-Scholes PDE when a foreign currency is involved

I revise and refine work from *Black Scholes PDEs (cross-currency) 1-6-99.nb.*

We develop an equation for a contingent claim F(S, X, t) where where X is the exchange rate (units of domestic currency per unit of foreign currency). We assume the stochastic processes dzS and dzX are correlated to degree ρ.

We consider a portfolio consisting of the contingent claim and hedges in both the underlying and the exchange rate

$$F[S, X, t] + \Delta S\, X\, S + \Delta X\, X$$

The portfolio value is measured in the domestic currency while the underlying is denominated in the foreign currency. The exchange rate is

$$X = \frac{\text{units of domestic currency}}{\text{units of foreign currency}}$$

Assume that these underliers follows a drifted geometric Brownian motion stochastic processes

In[5]:=
```
w1["processes"] = {sde[S] = dS/S == μS dt + σS dzS,
  sde[X] = dX/X == μX dt + σX dzX}
```

Out[5]=

$$\left\{\frac{dS}{S} == dt\,\mu S + dzS\,\sigma S,\ \frac{dX}{X} == dt\,\mu X + dzX\,\sigma X\right\}$$

We anticipate the hedge portfolio will consist of the contingent claim, some amount of the foreign equity, and some amount of the foreign currency.

In[6]:=
```
w1[1] = P[S, X, t] == F[S, X, t] + ΔS X S + ΔX X
```

Out[6]=

$$P[S, X, t] == S\,X\,\Delta S + X\,\Delta X + F[S, X, t]$$

Expand in a power series

In[7]:=
```
w1[2] =
 P[S, X, t] == Series[w1[1][[2]], {S, S0, 2}, {X, X0, 2}, {t, t0, 1}] // Normal
```

Out[7]=

$$\begin{aligned}
P[S, X, t] ==\ & S0\,X0\,\Delta S + X0\,\Delta X + F[S0, X0, t0] + (t - t0)\,F^{(0,0,1)}[S0, X0, t0] + \\
& (X - X0)\left(S0\,\Delta S + \Delta X + F^{(0,1,0)}[S0, X0, t0] + (t - t0)\,F^{(0,1,1)}[S0, X0, t0]\right) + \\
& (X - X0)^2\left(\frac{1}{2}F^{(0,2,0)}[S0, X0, t0] + \frac{1}{2}(t - t0)\,F^{(0,2,1)}[S0, X0, t0]\right) + \\
& (S - S0)\Bigg(X0\,\Delta S + F^{(1,0,0)}[S0, X0, t0] + (t - t0)\,F^{(1,0,1)}[S0, X0, t0] + \\
& \quad (X - X0)\left(\Delta S + F^{(1,1,0)}[S0, X0, t0] + (t - t0)\,F^{(1,1,1)}[S0, X0, t0]\right) + \\
& \quad (X - X0)^2\left(\frac{1}{2}F^{(1,2,0)}[S0, X0, t0] + \frac{1}{2}(t - t0)\,F^{(1,2,1)}[S0, X0, t0]\right)\Bigg) + \\
& (S - S0)^2\Bigg(\frac{1}{2}F^{(2,0,0)}[S0, X0, t0] + \frac{1}{2}(t - t0)\,F^{(2,0,1)}[S0, X0, t0] + \\
& \quad (X - X0)\left(\frac{1}{2}F^{(2,1,0)}[S0, X0, t0] + \frac{1}{2}(t - t0)\,F^{(2,1,1)}[S0, X0, t0]\right) + \\
& \quad (X - X0)^2\left(\frac{1}{4}F^{(2,2,0)}[S0, X0, t0] + \frac{1}{4}(t - t0)\,F^{(2,2,1)}[S0, X0, t0]\right)\Bigg)
\end{aligned}$$

In[8]:=
```
w1[3] = w1[2] /. {P[S, X, t] → dP + F[S0, X0, t0] + ΔS X0 S0 + ΔX X0 ,
    S → dS + S0, X → dX + X0, t → dt + t0} /. Equal → Subtract
```

Out[8]=

$$dP - dt\, F^{(0,0,1)}[S0, X0, t0] - dX \left(S0\, \Delta S + \Delta X + F^{(0,1,0)}[S0, X0, t0] + dt\, F^{(0,1,1)}[S0, X0, t0]\right) - dX^2 \left(\frac{1}{2} F^{(0,2,0)}[S0, X0, t0] + \frac{1}{2} dt\, F^{(0,2,1)}[S0, X0, t0]\right) - dS \Big(X0\, \Delta S + F^{(1,0,0)}[S0, X0, t0] + dt\, F^{(1,0,1)}[S0, X0, t0] + dX \left(\Delta S + F^{(1,1,0)}[S0, X0, t0] + dt\, F^{(1,1,1)}[S0, X0, t0]\right) + dX^2 \left(\frac{1}{2} F^{(1,2,0)}[S0, X0, t0] + \frac{1}{2} dt\, F^{(1,2,1)}[S0, X0, t0]\right)\Big) - dS^2 \Big(\frac{1}{2} F^{(2,0,0)}[S0, X0, t0] + \frac{1}{2} dt\, F^{(2,0,1)}[S0, X0, t0] + dX \left(\frac{1}{2} F^{(2,1,0)}[S0, X0, t0] + \frac{1}{2} dt\, F^{(2,1,1)}[S0, X0, t0]\right) + dX^2 \left(\frac{1}{4} F^{(2,2,0)}[S0, X0, t0] + \frac{1}{4} dt\, F^{(2,2,1)}[S0, X0, t0]\right)\Big)$$

In[9]:=
```
w1[4] = w1[3] /. {Solve[sde[S], dS][[1, 1]], Solve[sde[X], dX][[1, 1]]}
```

Out[9]=

$$dP - dt\, F^{(0,0,1)}[S0, X0, t0] - X\, (dt\, \mu X + dzX\, \sigma X) \left(S0\, \Delta S + \Delta X + F^{(0,1,0)}[S0, X0, t0] + dt\, F^{(0,1,1)}[S0, X0, t0]\right) - X^2 (dt\, \mu X + dzX\, \sigma X)^2 \left(\frac{1}{2} F^{(0,2,0)}[S0, X0, t0] + \frac{1}{2} dt\, F^{(0,2,1)}[S0, X0, t0]\right) - S\, (dt\, \mu S + dzS\, \sigma S) \Big(X0\, \Delta S + F^{(1,0,0)}[S0, X0, t0] + dt\, F^{(1,0,1)}[S0, X0, t0] + X\, (dt\, \mu X + dzX\, \sigma X) \left(\Delta S + F^{(1,1,0)}[S0, X0, t0] + dt\, F^{(1,1,1)}[S0, X0, t0]\right) + X^2 (dt\, \mu X + dzX\, \sigma X)^2 \left(\frac{1}{2} F^{(1,2,0)}[S0, X0, t0] + \frac{1}{2} dt\, F^{(1,2,1)}[S0, X0, t0]\right)\Big) - S^2 (dt\, \mu S + dzS\, \sigma S)^2 \Big(\frac{1}{2} F^{(2,0,0)}[S0, X0, t0] + \frac{1}{2} dt\, F^{(2,0,1)}[S0, X0, t0] + X\, (dt\, \mu X + dzX\, \sigma X) \left(\frac{1}{2} F^{(2,1,0)}[S0, X0, t0] + \frac{1}{2} dt\, F^{(2,1,1)}[S0, X0, t0]\right) + X^2 (dt\, \mu X + dzX\, \sigma X)^2 \left(\frac{1}{4} F^{(2,2,0)}[S0, X0, t0] + \frac{1}{4} dt\, F^{(2,2,1)}[S0, X0, t0]\right)\Big)$$

Introduce the diffusive ordering consistent with Ito

In[10]:=

```
w1[5] =
 w1[4] /. {dt -> ϵ dt , dzS → ϵ^(1/2) dzS , dzX → ϵ^(1/2) dzX} /. {S0 → S, X0 → X, t0 → t}
```

Out[10]=

$$dP - dt\,\epsilon\,F^{(0,0,1)}[S,X,t] - X\left(dt\,\epsilon\,\mu X + dzX\sqrt{\epsilon}\,\sigma X\right)\left(S\,\Delta S + \Delta X + F^{(0,1,0)}[S,X,t] + dt\,\epsilon\,F^{(0,1,1)}[S,X,t]\right) - X^2\left(dt\,\epsilon\,\mu X + dzX\sqrt{\epsilon}\,\sigma X\right)^2\left(\frac{1}{2}F^{(0,2,0)}[S,X,t] + \frac{1}{2}dt\,\epsilon\,F^{(0,2,1)}[S,X,t]\right) - S\left(dt\,\epsilon\,\mu S + dzS\sqrt{\epsilon}\,\sigma S\right)\left(X\,\Delta S + F^{(1,0,0)}[S,X,t] + dt\,\epsilon\,F^{(1,0,1)}[S,X,t] + X\left(dt\,\epsilon\,\mu X + dzX\sqrt{\epsilon}\,\sigma X\right)\left(\Delta S + F^{(1,1,0)}[S,X,t] + dt\,\epsilon\,F^{(1,1,1)}[S,X,t]\right) + X^2\left(dt\,\epsilon\,\mu X + dzX\sqrt{\epsilon}\,\sigma X\right)^2\left(\frac{1}{2}F^{(1,2,0)}[S,X,t] + \frac{1}{2}dt\,\epsilon\,F^{(1,2,1)}[S,X,t]\right)\right) - S^2\left(dt\,\epsilon\,\mu S + dzS\sqrt{\epsilon}\,\sigma S\right)^2\left(\frac{1}{2}F^{(2,0,0)}[S,X,t] + \frac{1}{2}dt\,\epsilon\,F^{(2,0,1)}[S,X,t] + X\left(dt\,\epsilon\,\mu X + dzX\sqrt{\epsilon}\,\sigma X\right)\left(\frac{1}{2}F^{(2,1,0)}[S,X,t] + \frac{1}{2}dt\,\epsilon\,F^{(2,1,1)}[S,X,t]\right) + X^2\left(dt\,\epsilon\,\mu X + dzX\sqrt{\epsilon}\,\sigma X\right)^2\left(\frac{1}{4}F^{(2,2,0)}[S,X,t] + \frac{1}{4}dt\,\epsilon\,F^{(2,2,1)}[S,X,t]\right)\right)$$

Introduce the Ito ordering and truncate the expansion

In[11]:=

```
w1[6] = ExpandAll[w1[5]] /. ϵ^(n_/;n>1) -> 0 /. ϵ -> 1 /.
   {dzS^2 → dt, dzX^2 → dt, dzS dzX → ρ dt}
```

Out[11]=

$$dP - dt\,S\,X\,\Delta S\,\mu S - dt\,S\,X\,\Delta S\,\mu X - dt\,X\,\Delta X\,\mu X - dzS\,S\,X\,\Delta S\,\sigma S - dzX\,S\,X\,\Delta S\,\sigma X - dzX\,X\,\Delta X\,\sigma X - dt\,S\,X\,\Delta S\,\rho\,\sigma S\,\sigma X - dt\,F^{(0,0,1)}[S,X,t] - dt\,X\,\mu X\,F^{(0,1,0)}[S,X,t] - dzX\,X\,\sigma X\,F^{(0,1,0)}[S,X,t] - \frac{1}{2}dt\,X^2\,\sigma X^2\,F^{(0,2,0)}[S,X,t] - dt\,S\,\mu S\,F^{(1,0,0)}[S,X,t] - dzS\,S\,\sigma S\,F^{(1,0,0)}[S,X,t] - dt\,S\,X\,\rho\,\sigma S\,\sigma X\,F^{(1,1,0)}[S,X,t] - \frac{1}{2}dt\,S^2\,\sigma S^2\,F^{(2,0,0)}[S,X,t]$$

In[12]:=

```
w1[7] = Collect[w1[6], {dt, dzS, dzX}]
```

Out[12]=

$$dP + dzX\left(-S\,X\,\Delta S\,\sigma X - X\,\Delta X\,\sigma X - X\,\sigma X\,F^{(0,1,0)}[S,X,t]\right) + dzS\left(-S\,X\,\Delta S\,\sigma S - S\,\sigma S\,F^{(1,0,0)}[S,X,t]\right) + dt\left(-S\,X\,\Delta S\,\mu S - S\,X\,\Delta S\,\mu X - X\,\Delta X\,\mu X - S\,X\,\Delta S\,\rho\,\sigma S\,\sigma X - F^{(0,0,1)}[S,X,t] - X\,\mu X\,F^{(0,1,0)}[S,X,t] - \frac{1}{2}X^2\,\sigma X^2\,F^{(0,2,0)}[S,X,t] - S\,\mu S\,F^{(1,0,0)}[S,X,t] - S\,X\,\rho\,\sigma S\,\sigma X\,F^{(1,1,0)}[S,X,t] - \frac{1}{2}S^2\,\sigma S^2\,F^{(2,0,0)}[S,X,t]\right)$$

The hedge parameters ΔS and ΔX can now be determined.

In[13]:=
```
w1[8] = {Solve[Coefficient[w1[7], dzS] == 0, ΔS][[1, 1]],
  Solve[Coefficient[w1[7], dzX] == 0, ΔX][[1, 1]]}
```

Out[13]=

$$\left\{\Delta S \to -\frac{F^{(1,0,0)}[S, X, t]}{X}, \Delta X \to -S\,\Delta S - F^{(0,1,0)}[S, X, t]\right\}$$

In[14]:=
```
w1[8] =
 Solve[{Coefficient[w1[7], dzS] == 0, Coefficient[w1[7], dzX] == 0}, {ΔS, ΔX}][[
   1]] // Expand
```

Out[14]=

$$\left\{\Delta S \to -\frac{F^{(1,0,0)}[S, X, t]}{X}, \Delta X \to -F^{(0,1,0)}[S, X, t] + \frac{S\,F^{(1,0,0)}[S, X, t]}{X}\right\}$$

Use these hedges to eliminate risk form the portfolio, i.e., cancel the coefficient of dzS and dzX.

In[15]:=
```
w1[9] = w1[7] /. w1[8]
```

Out[15]=

$$dP + dzX\left(-X\,\sigma X\,F^{(0,1,0)}[S, X, t] + S\,\sigma X\,F^{(1,0,0)}[S, X, t] - X\,\sigma X\left(-F^{(0,1,0)}[S, X, t] + \frac{S\,F^{(1,0,0)}[S, X, t]}{X}\right)\right) + dt\left(-F^{(0,0,1)}[S, X, t] - X\,\mu X\,F^{(0,1,0)}[S, X, t] - \frac{1}{2}X^2\,\sigma X^2\,F^{(0,2,0)}[S, X, t] + S\,\mu X\,F^{(1,0,0)}[S, X, t] + S\,\rho\,\sigma S\,\sigma X\,F^{(1,0,0)}[S, X, t] - X\,\mu X\left(-F^{(0,1,0)}[S, X, t] + \frac{S\,F^{(1,0,0)}[S, X, t]}{X}\right) - S\,X\,\rho\,\sigma S\,\sigma X\,F^{(1,1,0)}[S, X, t] - \frac{1}{2}S^2\,\sigma S^2\,F^{(2,0,0)}[S, X, t]\right)$$

The hedge portfolio is denominated in the domestic currency and should grow at the risk free rate in the domestic currency

$$dP = dt\,P\,rd$$

However, the portfolio was hedged by taking a position in the foreign currency. So the actual risk free appreciation of the portfolio is

$$dP = dt\,P\,rd + dt\,\Delta X\,rf\,X$$

In[16]:=
```
w1[10] = w1[9] /. dP → (F[S, X, t] + ΔS X S + ΔX X) rd dt + ΔX X rf dt
```

Out[16]=

$$dt\, rf\, X\, \Delta X + dt\, rd\, (S\, X\, \Delta S + X\, \Delta X + F[S, X, t]) + dzX\left(-X\, \sigma X\, F^{(0,1,0)}[S, X, t] + S\, \sigma X\, F^{(1,0,0)}[S, X, t] - X\, \sigma X\left(-F^{(0,1,0)}[S, X, t] + \frac{S\, F^{(1,0,0)}[S, X, t]}{X}\right)\right) + dt\left(-F^{(0,0,1)}[S, X, t] - X\, \mu X\, F^{(0,1,0)}[S, X, t] - \frac{1}{2} X^2\, \sigma X^2\, F^{(0,2,0)}[S, X, t] + S\, \mu X\, F^{(1,0,0)}[S, X, t] + S\, \rho\, \sigma S\, \sigma X\, F^{(1,0,0)}[S, X, t] - X\, \mu X\left(-F^{(0,1,0)}[S, X, t] + \frac{S\, F^{(1,0,0)}[S, X, t]}{X}\right) - S\, X\, \rho\, \sigma S\, \sigma X\, F^{(1,1,0)}[S, X, t] - \frac{1}{2} S^2\, \sigma S^2\, F^{(2,0,0)}[S, X, t]\right)$$

We introduce the explicit forms for the hedge parameters and eliminate the common factor dt by setting it to unity.

In[17]:=
```
w1[11] = w1[10] /. w1[8] /. dt → 1 // Expand
```

Out[17]=

$$rd\, F[S, X, t] - F^{(0,0,1)}[S, X, t] - rd\, X\, F^{(0,1,0)}[S, X, t] - rf\, X\, F^{(0,1,0)}[S, X, t] - \frac{1}{2} X^2\, \sigma X^2\, F^{(0,2,0)}[S, X, t] + rf\, S\, F^{(1,0,0)}[S, X, t] + S\, \rho\, \sigma S\, \sigma X\, F^{(1,0,0)}[S, X, t] - S\, X\, \rho\, \sigma S\, \sigma X\, F^{(1,1,0)}[S, X, t] - \frac{1}{2} S^2\, \sigma S^2\, F^{(2,0,0)}[S, X, t]$$

In[18]:=
```
w1[12] = Collect[w1[11], {F[S, X, t], F^(0,0,1)[S, X, t], F^(1,0,0)[S, X, t],
    F^(0,1,0)[S, X, t], F^(2,0,0)[S, X, t], F^(0,2,0)[S, X, t], F^(1,1,0)[S, X, t], S, X}]
```

Out[18]=

$$rd\, F[S, X, t] - F^{(0,0,1)}[S, X, t] + (-rd - rf)\, X\, F^{(0,1,0)}[S, X, t] - \frac{1}{2} X^2\, \sigma X^2\, F^{(0,2,0)}[S, X, t] + S\,(rf + \rho\, \sigma S\, \sigma X)\, F^{(1,0,0)}[S, X, t] - S\, X\, \rho\, \sigma S\, \sigma X\, F^{(1,1,0)}[S, X, t] - \frac{1}{2} S^2\, \sigma S^2\, F^{(2,0,0)}[S, X, t]$$

Reverse the sign to obtain the standard form

In[19]:=
```
w1[13] = -w1[12] == 0
```

Out[19]=

$$-rd\, F[S, X, t] + F^{(0,0,1)}[S, X, t] - (-rd - rf)\, X\, F^{(0,1,0)}[S, X, t] + \frac{1}{2} X^2\, \sigma X^2\, F^{(0,2,0)}[S, X, t] - S\,(rf + \rho\, \sigma S\, \sigma X)\, F^{(1,0,0)}[S, X, t] + S\, X\, \rho\, \sigma S\, \sigma X\, F^{(1,1,0)}[S, X, t] + \frac{1}{2} S^2\, \sigma S^2\, F^{(2,0,0)}[S, X, t] == 0$$

which is the desired PDE.

2 Specialization to Quanto option

A commonly occurring option involving a foreign currency is the quanto option, which is an option on a foreign stock with a strike price denominated in the domestic currency. The terminal payoff is

F[S, X, T] == Max[S - K, 0]

Note that this has no explicit dependence on the exchange rate X.

In[20]:=
```
w2[1] = w1[13] /. F → ((F[#1, #3]) &)
```

Out[20]=

$$-rd\,F[S,t]+F^{(0,1)}[S,t]-S\left(rf+\rho\,\sigma S\,\sigma X\right)F^{(1,0)}[S,t]+\frac{1}{2}S^2\,\sigma S^2\,F^{(2,0)}[S,t]==0$$

From *Black Scholes PDE Derivation 12-31-10.nb*, the classical BS PDE is

$$-r\,F[S,t]+(-q+r)\,S\,F_S[S,t]+\frac{1}{2}S^2\,\sigma^2\,F_{SS}[S,t]+F_t[S,t]==0$$

The only difference between these two forms is the coefficient of $F^{(1,0)}[S,t]$. We identify

In[21]:=
```
w2[2] = rd - q == -(rf + ρ σS σX)
```

Out[21]=

$$-q+rd==-rf-\rho\,\sigma S\,\sigma X$$

In[22]:=
```
w2[3] = Solve[w2[2], q][[1, 1]]
```

Out[22]=

$$q\to rd+rf+\rho\,\sigma S\,\sigma X$$

So, the fair value of the payoff of a quanto option is the same as a plain vanilla option provided that the dividend is expressed as above.