Black Scholes PDE Derivation for two underliers 01-02-11

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Initialization: Be sure the files *NTGStylesheet2.nb* and *NTGUtilityFunctions.m* is are in the same directory as that from which this notebook was loaded. Then execute the cell immediately below by mousing left on the cell bar to the right of that cell and then typing “shift” + “enter”. Respond “Yes” in response to the query to evaluate initialization cells.

In[23]:=

```
SetDirectory[NotebookDirectory[]];
(* set directory where source files are located *)
SetOptions[EvaluationNotebook[], (* load the StyleSheet *)
 StyleDefinitions → Get["NTGStylesheet2.nb"]];
Get["NTGUtilityFunctions.m"]; (* Load utilities package *)
```

1 Derivation of the Black-Scholes PDE for two underliers

I revise and refine work from *Black Scholes PDEs (exchange) 4-6-99.nb*

The following derivation extends Black Scholes PDE Derivation 12-31-10.nb to contingent claims that depend on two underliers S1 and S2

Assume that these underliers follows a drifted geometric Brownian motion stochastic processes

In[25]:=

```
{sde[S1] = dS1 / S1 == μ1 dt + σ1 dz1,
 sde[S2] = dS2 / S2 == μ2 dt + σ2 dz2}
```

Out[25]=

$$\left\{\frac{dS1}{S1} == dt\,\mu1 + dz1\,\sigma1,\ \frac{dS2}{S2} == dt\,\mu2 + dz2\,\sigma2\right\}$$

We perform a power series expansion to develop an equation for a contingent claim F(S1, S2, t).

In[26]:=
```
w1[1] = F[S1, S2, t] ==
   Series[F[S1, S2, t], {S1, S10, 2}, {S2, S20, 2}, {t, t0, 1}] // Normal
```

Out[26]=

$$F[S1, S2, t] == F[S10, S20, t0] + (t - t0)\, F^{(0,0,1)}[S10, S20, t0] + (S2 - S20)\left(F^{(0,1,0)}[S10, S20, t0] + (t - t0)\, F^{(0,1,1)}[S10, S20, t0]\right) + (S2 - S20)^2\left(\frac{1}{2} F^{(0,2,0)}[S10, S20, t0] + \frac{1}{2}(t - t0)\, F^{(0,2,1)}[S10, S20, t0]\right) + (S1 - S10)\left(F^{(1,0,0)}[S10, S20, t0] + (t - t0)\, F^{(1,0,1)}[S10, S20, t0] + (S2 - S20)\left(F^{(1,1,0)}[S10, S20, t0] + (t - t0)\, F^{(1,1,1)}[S10, S20, t0]\right) + (S2 - S20)^2\left(\frac{1}{2} F^{(1,2,0)}[S10, S20, t0] + \frac{1}{2}(t - t0)\, F^{(1,2,1)}[S10, S20, t0]\right)\right) + (S1 - S10)^2\left(\frac{1}{2} F^{(2,0,0)}[S10, S20, t0] + \frac{1}{2}(t - t0)\, F^{(2,0,1)}[S10, S20, t0] + (S2 - S20)\left(\frac{1}{2} F^{(2,1,0)}[S10, S20, t0] + \frac{1}{2}(t - t0)\, F^{(2,1,1)}[S10, S20, t0]\right) + (S2 - S20)^2\left(\frac{1}{4} F^{(2,2,0)}[S10, S20, t0] + \frac{1}{4}(t - t0)\, F^{(2,2,1)}[S10, S20, t0]\right)\right)$$

In[27]:=
```
w1[2] = w1[1] /. {F[S1, S2, t] → dF + F[S10, S20, t0],
    S1 → dS1 + S10, S2 → dS2 + S20, t → dt + t0} /. Equal → Subtract
```

Out[27]=

$$dF - dt\, F^{(0,0,1)}[S10, S20, t0] - dS2\left(F^{(0,1,0)}[S10, S20, t0] + dt\, F^{(0,1,1)}[S10, S20, t0]\right) - dS2^2\left(\frac{1}{2} F^{(0,2,0)}[S10, S20, t0] + \frac{1}{2} dt\, F^{(0,2,1)}[S10, S20, t0]\right) - dS1\left(F^{(1,0,0)}[S10, S20, t0] + dt\, F^{(1,0,1)}[S10, S20, t0] + dS2\left(F^{(1,1,0)}[S10, S20, t0] + dt\, F^{(1,1,1)}[S10, S20, t0]\right) + dS2^2\left(\frac{1}{2} F^{(1,2,0)}[S10, S20, t0] + \frac{1}{2} dt\, F^{(1,2,1)}[S10, S20, t0]\right)\right) - dS1^2\left(\frac{1}{2} F^{(2,0,0)}[S10, S20, t0] + \frac{1}{2} dt\, F^{(2,0,1)}[S10, S20, t0] + dS2\left(\frac{1}{2} F^{(2,1,0)}[S10, S20, t0] + \frac{1}{2} dt\, F^{(2,1,1)}[S10, S20, t0]\right) + dS2^2\left(\frac{1}{4} F^{(2,2,0)}[S10, S20, t0] + \frac{1}{4} dt\, F^{(2,2,1)}[S10, S20, t0]\right)\right)$$

In[28]:=
```
w1[3] = w1[2] /. {Solve[sde[S1], dS1][[1, 1]], Solve[sde[S2], dS2][[1, 1]]}
```

Out[28]=

$$dF - dt\, F^{(0,0,1)}[S10, S20, t0] - S2\,(dt\,\mu2 + dz2\,\sigma2)\left(F^{(0,1,0)}[S10, S20, t0] + dt\, F^{(0,1,1)}[S10, S20, t0]\right) - S2^2\,(dt\,\mu2 + dz2\,\sigma2)^2\left(\frac{1}{2}F^{(0,2,0)}[S10, S20, t0] + \frac{1}{2}dt\, F^{(0,2,1)}[S10, S20, t0]\right) - S1\,(dt\,\mu1 + dz1\,\sigma1)\left(F^{(1,0,0)}[S10, S20, t0] + dt\, F^{(1,0,1)}[S10, S20, t0] + S2\,(dt\,\mu2 + dz2\,\sigma2)\left(F^{(1,1,0)}[S10, S20, t0] + dt\, F^{(1,1,1)}[S10, S20, t0]\right) + S2^2\,(dt\,\mu2 + dz2\,\sigma2)^2\left(\frac{1}{2}F^{(1,2,0)}[S10, S20, t0] + \frac{1}{2}dt\, F^{(1,2,1)}[S10, S20, t0]\right)\right) - S1^2\,(dt\,\mu1 + dz1\,\sigma1)^2\left(\frac{1}{2}F^{(2,0,0)}[S10, S20, t0] + \frac{1}{2}dt\, F^{(2,0,1)}[S10, S20, t0] + S2\,(dt\,\mu2 + dz2\,\sigma2)\left(\frac{1}{2}F^{(2,1,0)}[S10, S20, t0] + \frac{1}{2}dt\, F^{(2,1,1)}[S10, S20, t0]\right) + S2^2\,(dt\,\mu2 + dz2\,\sigma2)^2\left(\frac{1}{4}F^{(2,2,0)}[S10, S20, t0] + \frac{1}{4}dt\, F^{(2,2,1)}[S10, S20, t0]\right)\right)$$

Introduce the diffusive ordering consistent with Ito

In[29]:=
```
w1[4] = w1[3] /. {dt -> ϵ dt , dz1 → ϵ^(1/2) dz1 , dz2 → ϵ^(1/2) dz2} /.
  {S10 → S1, S20 → S2, t0 → t}
```

Out[29]=

$$dF - dt\,\epsilon\, F^{(0,0,1)}[S1, S2, t] - S2\left(dt\,\epsilon\,\mu2 + dz2\sqrt{\epsilon}\,\sigma2\right)\left(F^{(0,1,0)}[S1, S2, t] + dt\,\epsilon\, F^{(0,1,1)}[S1, S2, t]\right) - S2^2\left(dt\,\epsilon\,\mu2 + dz2\sqrt{\epsilon}\,\sigma2\right)^2\left(\frac{1}{2}F^{(0,2,0)}[S1, S2, t] + \frac{1}{2}dt\,\epsilon\, F^{(0,2,1)}[S1, S2, t]\right) - S1\left(dt\,\epsilon\,\mu1 + dz1\sqrt{\epsilon}\,\sigma1\right)\left(F^{(1,0,0)}[S1, S2, t] + dt\,\epsilon\, F^{(1,0,1)}[S1, S2, t] + S2\left(dt\,\epsilon\,\mu2 + dz2\sqrt{\epsilon}\,\sigma2\right)\left(F^{(1,1,0)}[S1, S2, t] + dt\,\epsilon\, F^{(1,1,1)}[S1, S2, t]\right) + S2^2\left(dt\,\epsilon\,\mu2 + dz2\sqrt{\epsilon}\,\sigma2\right)^2\left(\frac{1}{2}F^{(1,2,0)}[S1, S2, t] + \frac{1}{2}dt\,\epsilon\, F^{(1,2,1)}[S1, S2, t]\right)\right) - S1^2\left(dt\,\epsilon\,\mu1 + dz1\sqrt{\epsilon}\,\sigma1\right)^2\left(\frac{1}{2}F^{(2,0,0)}[S1, S2, t] + \frac{1}{2}dt\,\epsilon\, F^{(2,0,1)}[S1, S2, t] + S2\left(dt\,\epsilon\,\mu2 + dz2\sqrt{\epsilon}\,\sigma2\right)\left(\frac{1}{2}F^{(2,1,0)}[S1, S2, t] + \frac{1}{2}dt\,\epsilon\, F^{(2,1,1)}[S1, S2, t]\right) + S2^2\left(dt\,\epsilon\,\mu2 + dz2\sqrt{\epsilon}\,\sigma2\right)^2\left(\frac{1}{4}F^{(2,2,0)}[S1, S2, t] + \frac{1}{4}dt\,\epsilon\, F^{(2,2,1)}[S1, S2, t]\right)\right)$$

Introduce the Ito ordering and truncate the expansion

In[30]:=
```
w1[5] = ExpandAll[w1[4]] /. ϵ^n_/;n>1 -> 0 /. ϵ -> 1 /.
  {dz1^2 → dt, dz2^2 → dt, dz1 dz2 → ρ dt}
```

Out[30]=

$$dF - dt\,F^{(0,0,1)}[S1, S2, t] - dt\,S2\,\mu 2\,F^{(0,1,0)}[S1, S2, t] - dz2\,S2\,\sigma 2\,F^{(0,1,0)}[S1, S2, t] - \frac{1}{2}dt\,S2^2\,\sigma 2^2\,F^{(0,2,0)}[S1, S2, t] - dt\,S1\,\mu 1\,F^{(1,0,0)}[S1, S2, t] - dz1\,S1\,\sigma 1\,F^{(1,0,0)}[S1, S2, t] - dt\,S1\,S2\,\rho\,\sigma 1\,\sigma 2\,F^{(1,1,0)}[S1, S2, t] - \frac{1}{2}dt\,S1^2\,\sigma 1^2\,F^{(2,0,0)}[S1, S2, t]$$

Form the hedge portfolio P = F + Δ1 S1+ Δ2 S2. The rate of change of this portfolio under the same ordering is

In[31]:=
```
w1[6] = dP == dF + Δ1 dS1 + Δ2 dS2 /.
  {Solve[sde[S1], dS1][[1, 1]], Solve[sde[S2], dS2][[1, 1]]}
```

Out[31]=

$$dP == dF + S1\,\Delta 1\,(dt\,\mu 1 + dz1\,\sigma 1) + S2\,\Delta 2\,(dt\,\mu 2 + dz2\,\sigma 2)$$

In detail

In[32]:=
```
w1[7] = w1[6] /. Solve[w1[5] == 0, dF][[1, 1]] /. dt -> ϵ dt /.
    {dz1 → ϵ^(1/2) dz1, dz1 → ϵ^(1/2) dz1} /. ϵ^n_/;n>1 -> 0 /. ϵ -> 1
```

Out[32]=

$$dP == S1\,\Delta 1\,(dt\,\mu 1 + dz1\,\sigma 1) + S2\,\Delta 2\,(dt\,\mu 2 + dz2\,\sigma 2) + \frac{1}{2}\Big(2\,dt\,F^{(0,0,1)}[S1, S2, t] + 2\,dt\,S2\,\mu 2\,F^{(0,1,0)}[S1, S2, t] + 2\,dz2\,S2\,\sigma 2\,F^{(0,1,0)}[S1, S2, t] + dt\,S2^2\,\sigma 2^2\,F^{(0,2,0)}[S1, S2, t] + 2\,dt\,S1\,\mu 1\,F^{(1,0,0)}[S1, S2, t] + 2\,dz1\,S1\,\sigma 1\,F^{(1,0,0)}[S1, S2, t] + 2\,dt\,S1\,S2\,\rho\,\sigma 1\,\sigma 2\,F^{(1,1,0)}[S1, S2, t] + dt\,S1^2\,\sigma 1^2\,F^{(2,0,0)}[S1, S2, t]\Big)$$

or

In[33]:=
```
w1[8] = w1[7][[1]] == Collect[ExpandAll[w1[7][[2]]], {dt, dz1, dz2}]
```

Out[33]=

$$dP == dz2\,(S2\,\Delta 2\,\sigma 2 + S2\,\sigma 2\,F^{(0,1,0)}[S1, S2, t]) + dz1\,(S1\,\Delta 1\,\sigma 1 + S1\,\sigma 1\,F^{(1,0,0)}[S1, S2, t]) + dt\Big(S1\,\Delta 1\,\mu 1 + S2\,\Delta 2\,\mu 2 + F^{(0,0,1)}[S1, S2, t] + S2\,\mu 2\,F^{(0,1,0)}[S1, S2, t] + \frac{1}{2}S2^2\,\sigma 2^2\,F^{(0,2,0)}[S1, S2, t] + S1\,\mu 1\,F^{(1,0,0)}[S1, S2, t] + S1\,S2\,\rho\,\sigma 1\,\sigma 2\,F^{(1,1,0)}[S1, S2, t] + \frac{1}{2}S1^2\,\sigma 1^2\,F^{(2,0,0)}[S1, S2, t]\Big)$$

We see that the stochastic risk of this portfolio can be removed by choosing α such that the term multiplying dz vanishes

In[34]:=
```
{Coefficient[w1[8][[2]], dz1] == 0, Coefficient[w1[8][[2]], dz2] == 0}
```

Out[34]=

$$\{S1\,\Delta 1\,\sigma 1 + S1\,\sigma 1\,F^{(1,0,0)}[S1, S2, t] == 0,\ S2\,\Delta 2\,\sigma 2 + S2\,\sigma 2\,F^{(0,1,0)}[S1, S2, t] == 0\}$$

In[35]:=
```
w1[9] = {Solve[Coefficient[w1[8][[2]], dz1] == 0, Δ1] [[1, 1]],
  Solve[Coefficient[w1[8][[2]], dz2] == 0, Δ2] [[1, 1]]}
```

Out[35]=
$$\left\{\Delta1 \to -F^{(1,0,0)}[S1, S2, t],\ \Delta2 \to -F^{(0,1,0)}[S1, S2, t]\right\}$$

Then the hedge portfolio is

In[36]:=
```
w1[10] = w1[8] /. w1[9]
```

Out[36]=
$$dP == dt\left(F^{(0,0,1)}[S1, S2, t] + \frac{1}{2} S2^2\, \sigma2^2\, F^{(0,2,0)}[S1, S2, t] + S1\, S2\, \rho\, \sigma1\, \sigma2\, F^{(1,1,0)}[S1, S2, t] + \frac{1}{2} S1^2\, \sigma1^2\, F^{(2,0,0)}[S1, S2, t]\right)$$

As in the single underlier case, the portfolio does not depend on the drifts terms, The portfolio grows at the risk free rate, thus

In[37]:=
```
w1[11] = dP == (r F[S1, S2, t] + r Δ1 S1 + r Δ2 S2 ) dt
```

Out[37]=
$$dP == dt\left(r\, S1\, \Delta1 + r\, S2\, \Delta2 + r\, F[S1, S2, t]\right)$$

or

In[38]:=
```
w1[12] = w1[11] /. w1[9]
```

Out[38]=
$$dP == dt\left(r\, F[S1, S2, t] - r\, S2\, F^{(0,1,0)}[S1, S2, t] - r\, S1\, F^{(1,0,0)}[S1, S2, t]\right)$$

Combining these two expressions for dP

In[39]:=
```
w1[13] = w1[10] /. (w1[12] /. Equal → Rule)
```

Out[39]=
$$dt\left(r\, F[S1, S2, t] - r\, S2\, F^{(0,1,0)}[S1, S2, t] - r\, S1\, F^{(1,0,0)}[S1, S2, t]\right) == dt\left(F^{(0,0,1)}[S1, S2, t] + \frac{1}{2} S2^2\, \sigma2^2\, F^{(0,2,0)}[S1, S2, t] + S1\, S2\, \rho\, \sigma1\, \sigma2\, F^{(1,1,0)}[S1, S2, t] + \frac{1}{2} S1^2\, \sigma1^2\, F^{(2,0,0)}[S1, S2, t]\right)$$

or

In[40]:=
```
w1[14] = w1[13] /. Equal → Subtract /. dt → 1
```

Out[40]=
$$r\, F[S1, S2, t] - F^{(0,0,1)}[S1, S2, t] - r\, S2\, F^{(0,1,0)}[S1, S2, t] - \frac{1}{2} S2^2\, \sigma2^2\, F^{(0,2,0)}[S1, S2, t] - r\, S1\, F^{(1,0,0)}[S1, S2, t] - S1\, S2\, \rho\, \sigma1\, \sigma2\, F^{(1,1,0)}[S1, S2, t] - \frac{1}{2} S1^2\, \sigma1^2\, F^{(2,0,0)}[S1, S2, t]$$

Let's reverse the sign and form an equation.

In[41]:=
```
w1[15] = -w1[14] == 0
```

Out[41]=

$$-r F[S1, S2, t] + F^{(0,0,1)}[S1, S2, t] + r\, S2\, F^{(0,1,0)}[S1, S2, t] + \frac{1}{2} S2^2\, \sigma 2^2\, F^{(0,2,0)}[S1, S2, t] + r\, S1\, F^{(1,0,0)}[S1, S2, t] + S1\, S2\, \rho\, \sigma 1\, \sigma 2\, F^{(1,1,0)}[S1, S2, t] + \frac{1}{2} S1^2\, \sigma 1^2\, F^{(2,0,0)}[S1, S2, t] == 0$$

2 Application to Margrabe exchange option

The Margrabe exchange option is an option to exchange one stock for another.

In[42]:=
```
w2[1] = w1[15] /. F → ((F[#1/#2, #3]) &) // Expand
```

Out[42]=

$$-r F\left[\frac{S1}{S2}, t\right] + F^{(0,1)}\left[\frac{S1}{S2}, t\right] - \frac{S1\, \rho\, \sigma 1\, \sigma 2\, F^{(1,0)}\left[\frac{S1}{S2}, t\right]}{S2} + \frac{S1\, \sigma 2^2\, F^{(1,0)}\left[\frac{S1}{S2}, t\right]}{S2} + \frac{S1^2\, \sigma 1^2\, F^{(2,0)}\left[\frac{S1}{S2}, t\right]}{2\, S2^2} - \frac{S1^2\, \rho\, \sigma 1\, \sigma 2\, F^{(2,0)}\left[\frac{S1}{S2}, t\right]}{S2^2} + \frac{S1^2\, \sigma 2^2\, F^{(2,0)}\left[\frac{S1}{S2}, t\right]}{2\, S2^2} == 0$$

Introduce $X = \frac{S1}{S2}$

In[43]:=
```
w2[2] = w2[1] /. S1 → X S2
```

Out[43]=

$$-r F[X, t] + F^{(0,1)}[X, t] - X\, \rho\, \sigma 1\, \sigma 2\, F^{(1,0)}[X, t] + X\, \sigma 2^2\, F^{(1,0)}[X, t] + \frac{1}{2} X^2\, \sigma 1^2\, F^{(2,0)}[X, t] - X^2\, \rho\, \sigma 1\, \sigma 2\, F^{(2,0)}[X, t] + \frac{1}{2} X^2\, \sigma 2^2\, F^{(2,0)}[X, t] == 0$$

In[44]:=
```
w2[3] = Collect[w2[2], {X, X^2}]
```

Out[44]=

$$-r F[X, t] + F^{(0,1)}[X, t] + X\left(-\rho\, \sigma 1\, \sigma 2\, F^{(1,0)}[X, t] + \sigma 2^2\, F^{(1,0)}[X, t]\right) + X^2\left(\frac{1}{2} \sigma 1^2\, F^{(2,0)}[X, t] - \rho\, \sigma 1\, \sigma 2\, F^{(2,0)}[X, t] + \frac{1}{2} \sigma 2^2\, F^{(2,0)}[X, t]\right) == 0$$

which is the appropriate PDE for the Margrabe option.