BenderOrszag Ex4-5-6-7 p289 04-18-16

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Initialization: Be sure the files *NTGStylesheet2.nb* and *NTGUtilityFunctions.m* is are in the same directory as that from which this notebook was loaded. Then execute the cell immediately below by mousing left on the cell bar to the right of that cell and then typing “shift” + “enter”. Respond “Yes” in response to the query to evaluate initialization cells.

In[3]:=
```
SetDirectory[NotebookDirectory[]];
(* set directory where source files are located *)
SetOptions[EvaluationNotebook[], (* load the StyleSheet *)
 StyleDefinitions → Get["NTGStylesheet2.nb"]];
Get["NTGUtilityFunctions.m"]; (* Load utilities package *)
```

Original notebook was *BenderOrszag Ex4-5-6-7 p289 09-21-15*

Purpose

I work through Bender-Orszag examples steepest descent-ascent curves passing through a saddle point.

I will use the following “standard” representation in the following examination of saddle points.

$$\exp(k\,\rho(z)) = \exp(k\,(\rho(x + i\,y)\}) = \exp(k\,(\phi(x,\ y) + i\,\psi(x,\ y)) \qquad (1)$$

1 Example 4

$$F = \exp\left(k\,z^2\right)$$

In[5]:=
```
Clear[ρ];
ρ[z_] := z^2
```

The single extrema occurs at

In[7]:=
```
w1[2] = Solve[D[ρ[z], z] == 0, z][[1, 1]]
```

Out[7]=
```
z → 0
```

In[8]:=
```
w1[2] = ϕ == ComplexExpand[Re[ρ[x + I y]]]
```
Out[8]= $\phi == x^2 - y^2$

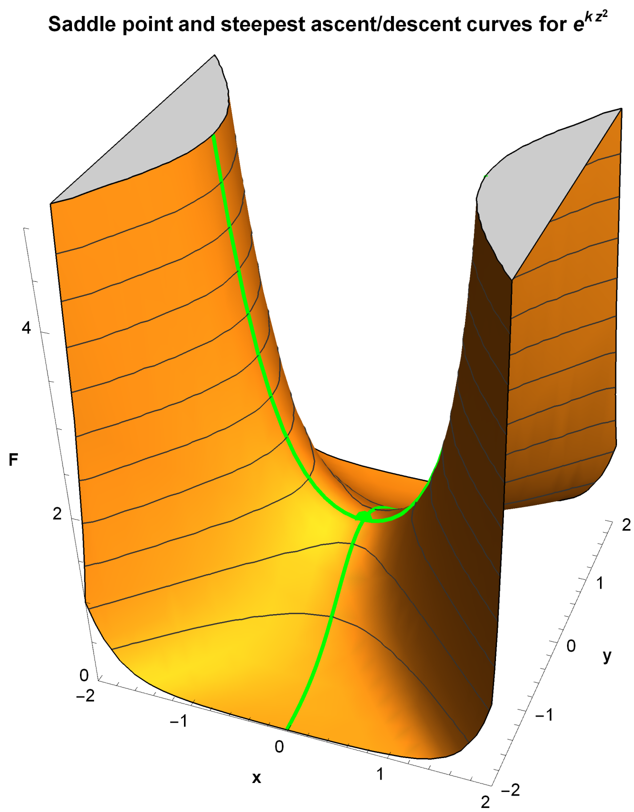
In[9]:=
```
w1[3] = ψ == ComplexExpand[Im[ρ[x + I y]]]
```
Out[9]= $\psi == 2\,x\,y$

Constant phase curves satisfy ψ be constant everywhere. Since $\psi(z = 0) = 0$, this means that either x = 0 everywhere or y = 0 everywhere.

In[10]:=
```
Module[{k = 1, X = 2, Y = 2, Z = 5, δF = 0.005,
  image = 300, saddlePoint, curve1, curve2, gCurves, gSurface, lab, F},
 F[z_, k_] := Exp[k z^2];

 saddlePoint = {GREEN, PointSize[0.03], Point[{0, 0, F[0, k] + δF}]};
 curve1 = {GREEN, Line@Table[{x, 0, F[x, k] + δF}, {x, -X, X, 0.1}]};
 curve2 = {GREEN, Line@Table[{0, y, F[I y, k] + δF}, {y, -Y, Y, 0.1}]};
 lab = Stl["Saddle point and steepest ascent/descent curves for e^(k z^2)"];
 gCurves = Graphics3D[{saddlePoint, curve1, curve2},
   PlotRange → {{-X, X}, {-Y, Y}, {0, Z}}, Boxed → False, PlotLabel → lab,
   Axes → Automatic, AxesLabel → {Stl["x"], Stl["y"], Stl["F"]}];
 gSurface = Plot3D[Abs[F[x + I y, k]], {x, -2, 2}, {y, -2, 2},
   ImageSize → image, MeshFunctions -> {#3 &}, Mesh → 10,
   Boxed → False, AxesLabel → {Stl["x"], Stl["y"], Stl["|f(z)|"]},
   PlotRange → {{-X, X}, {-Y, Y}, {0, Z}}];
 Show[{gCurves, gSurface}]]
```

Out[10]=



2 Example 5

$F = \exp(k\, i \cosh(z))$

In[17]:=
```
Clear[ρ];
ρ[z_] := I Cosh[z]
```

The extrema are at

In[19]:=
```
w2[1] = Solve[D[ρ[z], z] == 0, z][[1, 1]]
```

Out[19]=
```
z → ConditionalExpression[2 ⅈ π C[1], C[1] ∈ Integers]
```

There is an infinite series of extrema evenly spaced along the imaginary axis

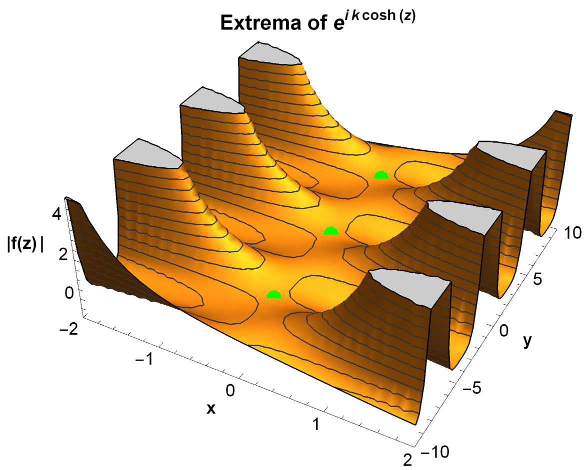
In[20]:=
```
w2[2] = w2[1] /. C[1] → Range[-2, 2]
```

Out[20]=
```
z → {-4 ⅈ π, -2 ⅈ π, 0, 2 ⅈ π, 4 ⅈ π}
```

In[22]:=
```
Module[{k = 1, X = 2, Y = 10, Z = 5, δF = 0.01,
  image = 300, extremaList, gPoints, gSurface, lab, F},
 F[z_, k_] := Exp[ k ρ[z]];

 extremaList = {GREEN, PointSize[0.03],
   Point[{0, Im[#], Abs[F[#, k]] + δF}] & /@ {-4 ⅈ π, -2 ⅈ π, 0, 2 ⅈ π, 4 ⅈ π}};
 lab = Stl["Extrema of e^(i k cosh (z))"];
 gPoints =
  Graphics3D[extremaList, PlotRange → {{-X, X}, {-Y, Y}, {0, Z}}, Axes → Automatic,
   AxesLabel → {Stl["x"], Stl["y"], Stl["|F|"]}, Boxed → False, PlotLabel → lab];
 gSurface = Plot3D[Abs[F[x + I y, k]], {x, -X, X}, {y, -Y, Y}, MeshFunctions -> {#3 &},
   Mesh → 10, Boxed → False, AxesLabel → {Stl["x"], Stl["y"], Stl["|f(z)|"]},
   PlotLabel → lab, PlotRange → {{-X, X}, {-Y, Y}, {-1, Z}}, ImageSize → 300];
 Show[{gSurface, gPoints}]]
```

Out[22]=



The symmetry is apparent so I focus on the extrema at z = 0

In[23]:=
```
w2[3] = Exp[ρ[z]] /. z → 0
```
Out[23]=
```
ⅇ^ⅈ
```

In[24]:=
```
w2[4] = ϕ == ComplexExpand[Re[ρ[ x + I y] ]]
```
Out[24]=
```
ϕ == -Sin[y] Sinh[x]
```

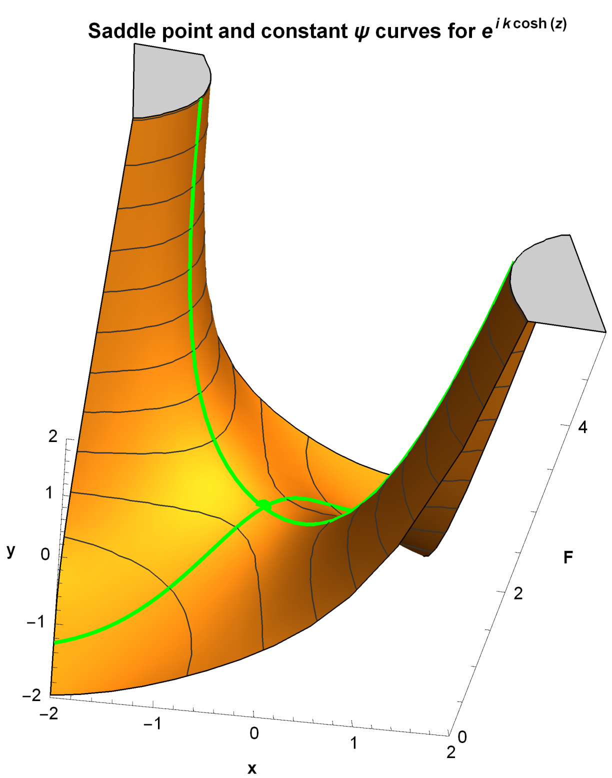
In[25]:=
```
w2[5] = ψ == ComplexExpand[Im[ρ[ x + I y] ]]
```
Out[25]=
```
ψ == Cos[y] Cosh[x]
```

So Im{ρ[t]} = i orψ = i determines the constant ψ curves

In[26]:=
```
w2[6] = w2[5][[2]] == 1
```
Out[26]=
```
Cos[y] Cosh[x] == 1
```

In[27]:=
```
w2[7] = Solve[w2[6], y] /. C[1] → 0
```
Out[27]=
```
{{y → -ArcCos[Sech[x]]}, {y → ArcCos[Sech[x]]}}
```

Constant phase curves satisfy ψ be constant everywhere.



3 Example 6

$f(z) = \exp(\rho(z)) = \exp(k\,(\sinh(z) - z))$

In[28]:=
```
Clear[ρ];
ρ[z_] := Sinh[z] - z
```

In[30]:=
```
w3[1] = Solve[D[ρ[z], z] == 0, z][[1, 1]]
```

Out[30]=
```
z → ConditionalExpression[2 ⅈ π C[1], C[1] ∈ Integers]
```

There is an infinite series of extrema evenly spaced along the imaginary axis

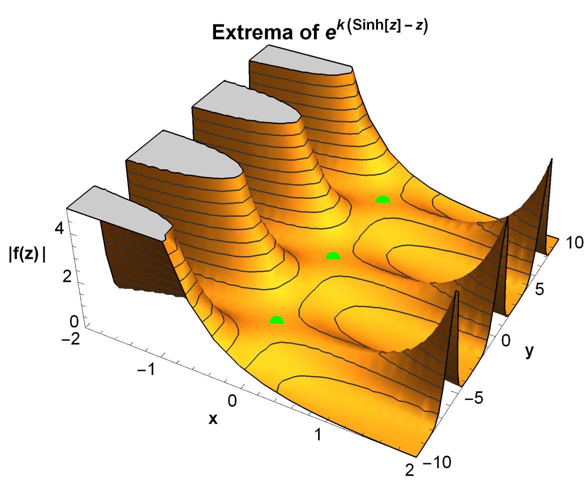
In[31]:=
```
w3[2] = w3[1] /. C[1] → Range[-2, 2]
```

Out[31]=
```
z → {-4 ⅈ π, -2 ⅈ π, 0, 2 ⅈ π, 4 ⅈ π}
```

In[32]:=
```
Module[{k = 1, X = 2, Y = 10, Z = 5, δF = 0.01,
   image = 300, extremaList, gPoints, gSurface, lab, F},
 F[z_, k_] := Exp[ k ρ[z]];

 extremaList = {GREEN, PointSize[0.03],
   Point[{0, Im[#], Abs[F[#, k]] + δF}] &/@ {-4 ⅈ π, -2 ⅈ π, 0, 2 ⅈ π, 4 ⅈ π}};
 lab = Stl["Extrema of e^(k (Sinh[z] - z))"];
 gPoints =
  Graphics3D[extremaList, PlotRange → {{-X, X}, {-Y, Y}, {0, Z}}, Axes → Automatic,
   AxesLabel → {Stl["x"], Stl["y"], Stl["|F|"]}, Boxed → False, PlotLabel → lab];
 gSurface = Plot3D[Abs[F[x + I y, k]], {x, -X, X}, {y, -Y, Y}, MeshFunctions -> {#3 &},
   Mesh → 10, Boxed → False, AxesLabel → {Stl["x"], Stl["y"], Stl["|f(z)|"]},
   PlotLabel → lab, PlotRange → {{-X, X}, {-Y, Y}, {0, Z}}, ImageSize → 300];
 Show[{gSurface, gPoints}]]
```

Out[32]=



Focus on the extrema at z = 0

In[33]:=
```
w3[3] = Exp[ρ[z]] /. z → 0
```
Out[33]=
```
1
```

In[34]:=
```
w3[4] = ϕ == ComplexExpand[Re[ρ[ x + I y] ]]
```
Out[34]=
```
ϕ == -x + Cos[y] Sinh[x]
```

In[35]:=
```
w3[5] = ψ == ComplexExpand[Im[ρ[ x + I y] ]]
```
Out[35]=
```
ψ == -y + Cosh[x] Sin[y]
```

Note that this is not a simple saddle point, ρ'' = 0 and ρ''' ≠ 0

In[36]:=
```
w3[6] = {D[ρ[z], {z, 2}], D[ρ[z], {z, 3}]} /. z → 0
```
Out[36]=
```
{0, 1}
```

Find expressions for the curves of steepest ascent and descent

In[37]:=
```
w3[7] = Exp[ρ[z]] /. z → 0
```
Out[37]=
```
1
```

In[38]:=
```
w3[4] = ϕ == ComplexExpand[Re[ρ[ x + I y] ]]
```
Out[38]=
```
ϕ == -x + Cos[y] Sinh[x]
```

In[39]:=
```
w3[5] = ψ == ComplexExpand[Im[ρ[ x + I y] ]]
```
Out[39]=
```
ψ == -y + Cosh[x] Sin[y]
```

The constant ψ curves are given by

In[40]:=
```
w3[7] = w3[5][[2]] == 0
```
Out[40]=
```
-y + Cosh[x] Sin[y] == 0
```

One way to approach this is to consider the case where y << 1

In[41]:=
```
w3[8] = Normal@Series[w3[7], {y, 0, 3}]
```
Out[41]=

$$y\left(-1+\mathrm{Cosh}[x]\right)-\frac{1}{6}y^3\,\mathrm{Cosh}[x]==0$$

In[42]:=
```
w3[9] = Solve[w3[8], y] // Simplify
```

Out[42]=

$\{\{y \to 0\}, \{y \to -\sqrt{6 - 6\,\mathrm{Sech}[x]}\}, \{y \to \sqrt{6 - 6\,\mathrm{Sech}[x]}\}\}$

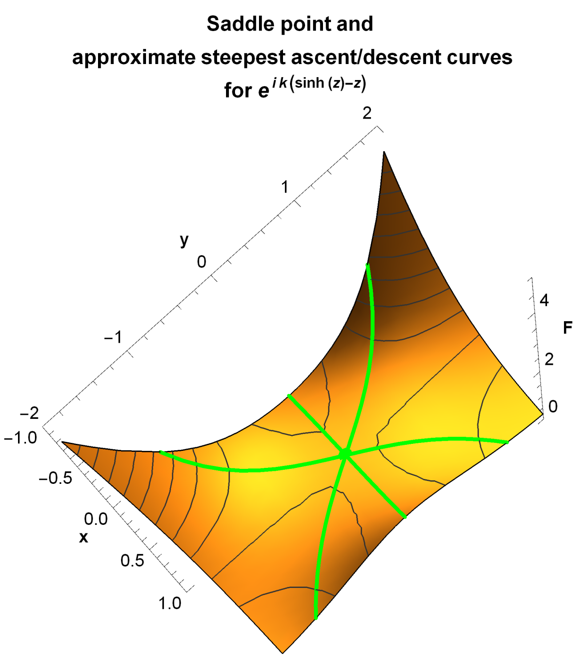
Constant phase curves satisfy ψ = constant everywhere.

In[43]:=

```
Module[{k = 1, X = 1, Y = 2, Z = 5, δF = 0.005, image = 300, saddlePoint,
  curve1, curve2, curve3, gCurves, gSurface, lab, F, y1, y2, y3},
 F[z_, k_] := Exp[ k (Sinh[z] - z)];
 y1[z_, k_] := 0;
 y2[z_, k_] := -√6 √(1 - Sech[x]) ;
 y3[z_, k_] := √6 √(1 - Sech[x]) ;

 saddlePoint = {GREEN, PointSize[0.03], Point[{0, 0, Abs@F[0, k] + δF}]};
 curve1 =
  {GREEN, Line@Table[{x, y1[x, k], Abs@F[x + I y1[x, k] , k] + δF}, {x, -X, X, 0.1}]};
 curve2 = {GREEN, Line@
    Table[{x, y2[x, k], Abs@F[x + I y2[x, k] , k] + δF}, {x, -X, X, 0.1}]};
 curve3 = {GREEN, Line@Table[{x, y3[x, k], Abs@F[x + I y3[x, k] , k] + δF},
     {x, -X, X, 0.1}]};
 lab = Stl["Saddle point and \napproximate steepest
     ascent/descent curves\n for e^(i k (sinh (z) - z))"];
 gCurves = Graphics3D[{saddlePoint, curve1, curve2, curve3},
   PlotRange → {{-X, X}, {-Y, Y}, {0, Z}}, Boxed → False, PlotLabel → lab,
   Axes → Automatic, AxesLabel → {Stl["x"], Stl["y"], Stl["F"]}];
 gSurface = Plot3D[Abs[F[x + I y, k]], {x, -X, X}, {y, -Y, Y},
   ImageSize → image, MeshFunctions -> {#3 &}, Mesh → 10,
   Boxed → False, AxesLabel → {Stl["x"], Stl["y"], Stl["|f(z)|"]},
   PlotRange → {{-X, X}, {-Y, Y}, {0, Z}}];
 Show[{gCurves, gSurface}]]
```

Out[43]=



4 Example 7

$$f(z) = \exp(\rho(z)) = \exp\left(k\left(\cosh(z) - z^2/2\right)\right)$$

In[44]:=
```
Clear[ρ];
ρ[z_] := Cosh[z] - z^2/2
```

Mathematica can’t find a closed form.

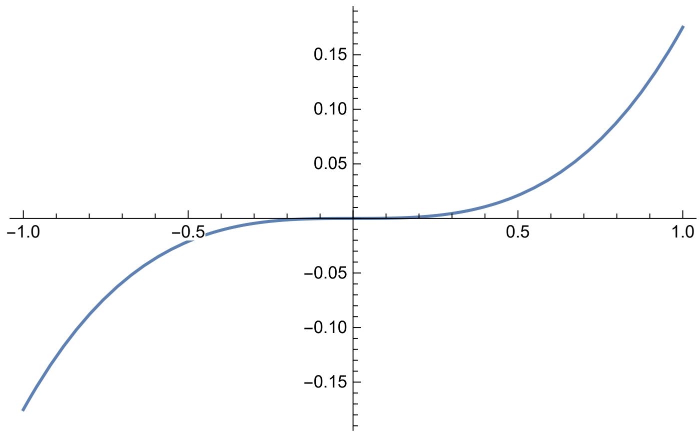
In[46]:=
```
w4[1] = Solve[D[ρ[z], z] == 0, z][[1, 1]]
```

Solve: This system cannot be solved with the methods available to Solve.

Out[46]=
```
-z + Sinh[z]
```

but visualization shows that

In[47]:=
```
Plot[-z + Sinh[z], {z, -1, 1}]
```

Out[47]=


In[48]:=
```
w4[2] = Normal@Series[ρ[z], {z, 0, 8}]
```

Out[48]=
$$1 + \frac{z^4}{24} + \frac{z^6}{720} + \frac{z^8}{40320}$$

This is a higher order saddle point, as the first four derivatives confirm

In[49]:=
```
w4[3] = (D[w4[2], {z, #}]) & /@ {1, 2, 3, 4} /. z → 0
```

Out[49]=
```
{0, 0, 0, 1}
```

Find expressions for the curves of steepest ascent and descent

In[50]:=
```
ρ[z]
```

Out[50]=
$$-\frac{z^2}{2}+\text{Cosh}[z]$$

In[51]:=
```
w4[4] = Exp[ρ[0]]
```

Out[51]=
$$e$$

In[52]:=
```
w4[5] = ϕ == ComplexExpand[Re[ρ[ x + I y] ]]
```

Out[52]=
$$\phi == -\frac{x^2}{2}+\frac{y^2}{2}+\text{Cos}[y]\,\text{Cosh}[x]$$

In[53]:=
```
w4[6] = ψ == ComplexExpand[Im[ρ[ x + I y] ]]
```

Out[53]=
$$\psi == -x\,y+\text{Sin}[y]\,\text{Sinh}[x]$$

The constant ψ curves are given by

In[54]:=
```
w4[7] = w4[6][[2]] == 0
```

Out[54]=
$$-x\,y+\text{Sin}[y]\,\text{Sinh}[x] == 0$$

This is transcendental and has no closed form solution

In[55]:=
```
w4[8] = Solve[w4[7], y]
```

Solve: This system cannot be solved with the methods available to Solve.

Out[55]=
$$\text{Solve}[-x\,y+\text{Sin}[y]\,\text{Sinh}[x] == 0,\ y]$$

So, I look for solutions near z = 0

In[56]:=
```
w4[9] = Normal@Series[w4[8][[1]], {y, 0, 3}, {x, 0, 3}]
```

Out[56]=
$$\frac{x^3\,y}{6}+\left(-\frac{x}{6}-\frac{x^3}{36}\right)y^3 == 0$$

In[57]:=
```
w4[10] = Solve[w4[9], y]
```

Out[57]=
$$\left\{\{y\to 0\},\ \left\{y\to -\frac{\sqrt{6}\,x}{\sqrt{6+x^2}}\right\},\ \left\{y\to \frac{\sqrt{6}\,x}{\sqrt{6+x^2}}\right\}\right\}$$

One of the solutions emerges along the y = 0 line so I have to solve for x = x(y).

In[58]:=
```
w4[11] = Solve[w4[9], x]
```

Out[58]=

$$\left\{\{x \to 0\},\ \left\{x \to -\frac{\sqrt{6}\ y}{\sqrt{6-y^2}}\right\},\ \left\{x \to \frac{\sqrt{6}\ y}{\sqrt{6-y^2}}\right\}\right\}$$

In[59]:=
```
Module[{k = 1, X = 2, Y = 2, Z = 5, δF = 0.005, image = 300, saddlePoint,
  curve1, curve2, curve3, curve4, gCurves, gSurface, lab, F, y1, y2, y3, x1},
 F[z_, k_] := Exp[k (Cosh[z] - z^2/2)];
 y1[z_, k_] := 0;
 y2[z_, k_] := -(Sqrt[6] x)/Sqrt[6 + x^2];
 y3[z_, k_] := (Sqrt[6] x)/Sqrt[6 + x^2];
 x1[z_, k_] := 0;

 saddlePoint = {GREEN, PointSize[0.03], Point[{0, 0, Abs@F[0, k] + δF}]};
 curve1 =
  {GREEN, Line@Table[{x, y1[x, k], Abs@F[x + I y1[x, k] , k] + δF}, {x, -X, X, 0.1}]};
 curve2 = {GREEN, Line@
    Table[{x, y2[x, k], Abs@F[x + I y2[x, k] , k] + δF}, {x, -X, X, 0.1}]};
 curve3 = {GREEN, Line@Table[{x, y3[x, k], Abs@F[x + I y3[x, k] , k] + δF},
     {x, -X, X, 0.1}]};
 curve4 = {GREEN, Line@Table[{x1[y, k], y, Abs@F[x1[y, k] + I y , k] + δF},
     {y, -Y, Y, 0.1}]};

 lab = Stl["Saddle point and \napproximate
     steepest ascent/descent curves\n for e^(i k (cosh (z) - z^2/2))"];
 gCurves = Graphics3D[{saddlePoint, curve1, curve2, curve3, curve4},
   PlotRange → {{-X, X}, {-Y, Y}, {0, Z}}, Boxed → False, PlotLabel → lab,
   Axes → Automatic, AxesLabel → {Stl["x"], Stl["y"], Stl["F"]}];
 gSurface = Plot3D[Abs[F[x + I y, k]], {x, -2, 2}, {y, -2, 2},
   ImageSize → image, MeshFunctions -> {#3 &}, Mesh → 10,
   Boxed → False, AxesLabel → {Stl["x"], Stl["y"], Stl["|f(z)|"]},
   PlotRange → {{-X, X}, {-Y, Y}, {0, Z}}];
 Show[{gCurves, gSurface}]]
```

Out[59]=

Saddle point and
approximate steepest ascent/descent curves
for $e^{i\,k\left(\cosh(z)-\frac{z^2}{2}\right)}$