

BenderOrszag Ex09 p294 04-30-16

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Initialization: Be sure the files *NTGStylesheet2.nb* and *NTGUtilityFunctions.m* is are in the same directory as that from which this notebook was loaded. Then execute the cell immediately below by mousing left on the cell bar to the right of that cell and then typing “shift” + “enter”. Respond “Yes” in response to the query to evaluate initialization cells.

In[5]:=

```
SetDirectory[NotebookDirectory[]];
(* set directory where source files are located *)
SetOptions[EvaluationNotebook[], (* load the StyleSheet *)
  StyleDefinitions -> Get["NTGStylesheet2.nb"]];
Get["NTGUtilityFunctions.m"]; (* Load utilities package *)
```

Original notebook — *BenderOrszag Ex10 p296 09-29-15 (accidentally overwritten)*

Purpose

Approximation of Gamma function using the saddle point method.

I Example 9

An integral representation of the Gamma function is

$$\frac{1}{\Gamma(k)} = \frac{1}{2\pi i} \int_C dz e^z z^{-k}$$

where C is a contour that wraps around the branch cut associated with z^{-k} .



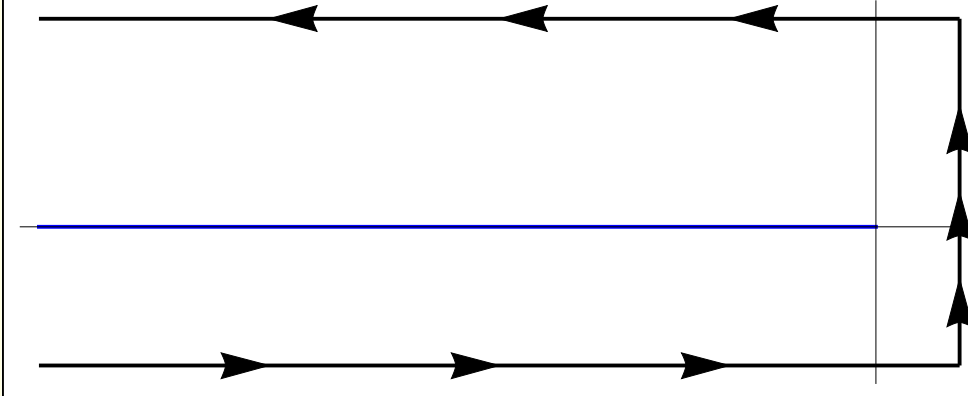
Just for fun, I check against a numerical valuation of the contour integral using a simple path of the form above.

In[7]:=

```
Module[{k = 5, a = 0.2, b = 0.3, xLo = -10, xHi = 1},
  NumericalContourIntegration[k, xLo, xHi, a, b]
```

k = 5, 1/Γ[k] = 0.0416667, Numerical Contour Integration = 0.0416667

Out[7]=



In[3]:=

```
Clear[NumericalContourIntegration];
NumericalContourIntegration[k_, xLo_, xHi_, a_, b_] :=
Module[{path, branchcut, C, numericalResult, lab},
  path[1] = NIntegrate[ $\frac{1}{2\pi i} \text{Exp}[z] z^{-k} dz /. \{z \rightarrow x - I a, dz \rightarrow 1\}, \{x, xLo, xHi\}];$ 
  path[2] = NIntegrate[ $\frac{1}{2\pi i} \text{Exp}[z] z^{-k} dz /. \{z \rightarrow xHi + I y, dz \rightarrow I\}, \{y, -a, b\}];$ 
  path[3] = NIntegrate[ $\frac{1}{2\pi i} \text{Exp}[z] z^{-k} dz /. \{z \rightarrow x + I b, dz \rightarrow 1\}, \{x, xHi, xLo\}];$ 
  numericalResult = Chop[path[1] + path[2] + path[3]];
  branchcut = {BLUE, Line[{{xLo, 0}, {0, 0}}]};
  C[1] =
    {BLACK, Arrowheads[{{0, 0.05, 0.05, 0.05, 0}}, Arrow[{{xLo, -a}, {xHi, -a}}]};
  C[2] = {BLACK, Arrowheads[{{0, 0.05, 0.05, 0.05, 0}},
    Arrow[{{xHi, -a}, {xHi, b}}]};
  C[3] = {BLACK, Arrowheads[{{0, 0.05, 0.05, 0.05, 0}}, Arrow[{{xHi, b}, {xLo, b}}]};
  lab = Stl@StringForm["k = ``, 1/Γ[k] = ``, Numerical Contour Integration = ``",
    k, numericalResult, N[1/Gamma[k]]];
  Graphics[{branchcut, C[1], C[2], C[3]}, Axes → Automatic,
    AspectRatio → 0.4, ImageSize → 500, Ticks → None, PlotLabel → lab]
```

2 Change of variable in preparation for the steepest descent analysis

The standard form I have been using for steepest descent problems is

$$\int dz e^{k\rho(z)}$$

But, in this case the integrand as defined is

$$\text{In[8]:= } w2[1] = \frac{1}{2 \pi i} \text{Exp}[z] z^{-k} dz$$

$$\text{Out[8]= } - \frac{i dz e^z z^{-k}}{2 \pi}$$

A change of variables achieves the desired form. First rewrite the power term

$$\text{In[9]:= } w2[2] = \frac{1}{2 \pi i} \text{Exp}[z - k \text{Log}[z]] dz$$

$$\text{Out[9]= } - \frac{i dz e^{z-k \text{Log}[z]}}{2 \pi}$$

Make the change of variables

$$\text{In[10]:= } w2[3] = w2[2] /. \{z \rightarrow k \xi, dz \rightarrow k d\xi\}$$

$$\text{Out[10]= } - \frac{i d\xi e^{k \xi - k \text{Log}[k \xi]} k}{2 \pi}$$

Expanding the log term and changing the dummy integration variable back to z yields

$$\text{In[11]:= } w2[4] = w2[3] /. \text{Log}[k \xi] \rightarrow \text{Log}[k] + \text{Log}[\xi] /. \{\xi \rightarrow z, d\xi \rightarrow dz\} // \text{PowerExpand}$$

$$\text{Out[11]= } - \frac{i dz e^{k z - k (\text{Log}[k] + \text{Log}[z])} k}{2 \pi}$$

This has the desired form. Since this transformation was so simple, I performed it by just typing the desired expressions. In Appendix A, I show how this could have been accomplished programmatically. The result of that operation was

$$\text{In[12]:= } w2[5] = - \frac{i d\xi e^{k (\xi - \text{Log}[\xi])} k^{1-k}}{2 \pi}$$

$$\text{Out[12]= } - \frac{i d\xi e^{k (\xi - \text{Log}[\xi])} k^{1-k}}{2 \pi}$$

3 Calculation of the saddle point and steepest descent curve

$$\text{In[13]:= } \text{Clear}[\rho];$$

$$\rho[z_]:= z - \text{Log}[z]$$

There is a single simple saddle point.

In[15]:= **w3[1] = Solve[D[ρ[z], z] == 0][[1]]**

Out[15]= {z → 1}

In[16]:= **w3[2] = φ == ComplexExpand[Re[ρ[x + I y]]]**

Out[16]= $\phi = x - \frac{1}{2} \text{Log}[x^2 + y^2]$

In[17]:= **w3[3] = ψ == ComplexExpand[Im[ρ[x + I y]]]**

Out[17]= $\psi = y - \text{Arg}[x + i y]$

In[18]:= **w3[4] = w3[3] /. Arg[x + i y] → ArcTan[$\frac{y}{\sqrt{x^2 + y^2}}$]**

Out[18]= $\psi = y - \text{ArcTan}\left[\frac{y}{\sqrt{x^2 + y^2}}\right]$

The constant ψ curves passing through the saddle point $z = 1$ is given by

In[19]:= **w3[5] = w3[4] /. ψ → Im[ρ[1]]**

Out[19]= $\theta = y - \text{ArcTan}\left[\frac{y}{\sqrt{x^2 + y^2}}\right]$

This can be solved for x.

In[20]:= **w3[6] = Solve[w3[5], x]**

Out[20]= $\left\{ \left\{ x \rightarrow -\text{Cot}[y] \sqrt{y^2 - y^2 \text{Tan}[y]^2} \right\}, \left\{ x \rightarrow \text{Cot}[y] \sqrt{y^2 - y^2 \text{Tan}[y]^2} \right\} \right\}$

Near the saddle point

In[21]:= **w3[7] = Normal@Series[{-Cot[y] $\sqrt{y^2 - y^2 \text{Tan}[y]^2}$, Cot[y] $\sqrt{y^2 - y^2 \text{Tan}[y]^2}$ }, {y, 0, 2}]**

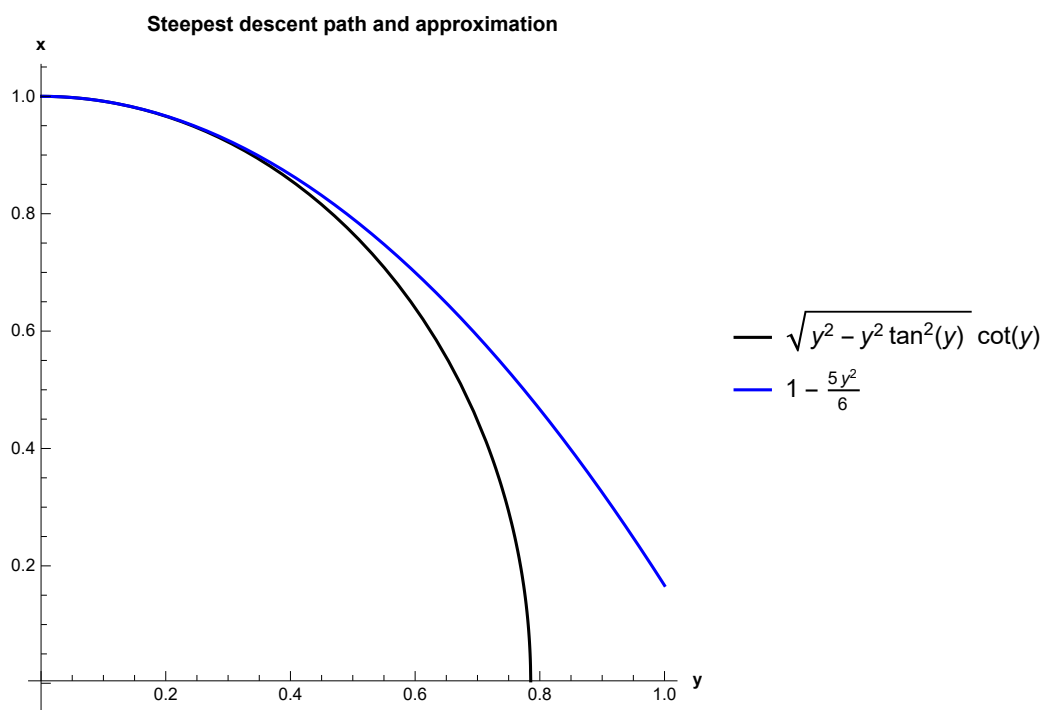
Out[21]= $\left\{ -1 + \frac{5y^2}{6}, 1 - \frac{5y^2}{6} \right\}$

In[22]:=

```
Module[{lab, F},
  F[y_] := Cot[y]  $\sqrt{y^2 - y^2 \tan^2[y]}$ ;

  lab = St1@
    StringForm["Steepest descent path and approximation", TraditionalForm[F[x, k]]];
  Plot[{F[y],  $1 - \frac{5y^2}{6}$ }, {y, 0.001, 1}, PlotStyle -> {Black, Blue}, PlotLegends -> Placed[
    {TraditionalForm[Cot[y]  $\sqrt{y^2 - y^2 \tan^2[y]}$ ], TraditionalForm[ $1 - \frac{5y^2}{6}$ ]}, Right],
    AxesLabel -> {St1["y"], St1["x"]}, PlotLabel -> lab,
    PlotRange -> All, AspectRatio -> 1]]
```

Out[22]=

Setting $y = 0$

In[23]:=

```
w3[7] /. y -> 0
```

Out[23]=

```
{-1, 1}
```

shows that the second root is the one passing through the saddle point $z = 1$.

In[24]:=

```

Module[{k = 2, X = 2, Y = 3, Z = 10, δF = 0.02, image = 400, lab,
  F, saddlePoint, curve, CurveXofY, SaddlePoint, gSurface, gCurves},
  F[z_, k_] := Exp[k ρ[z]];
  CurveXofY[fXofY_, {yStart_, yFin_, dy_}] :=
    {GREEN, Tooltip[Arrow@Table[{fXofY[y], y, Abs@F[fXofY[y] + I y, k] + δF},
      {y, yStart, yFin, dy}], StringForm["x = `", TraditionalForm[fXofY[y]]]}}];
  SaddlePoint[zSP_] := {DARKGREEN, PointSize[0.03], Tooltip[
    Point[{Re[zSP], Im[zSP], Abs@F[zSP, k] + δF}], StringForm["zSP = `", zSP]}}];

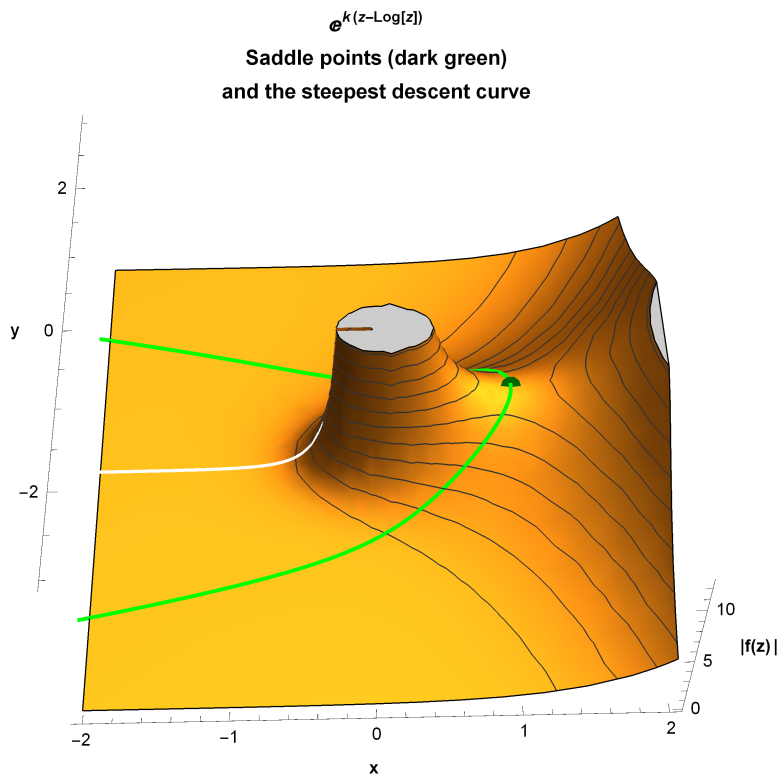
  saddlePoint[1] = SaddlePoint[1];
  curve[1] = CurveXofY[Function[{y}, 1 -  $\frac{5y^2}{6}$ ], {-Y, Y, 0.05}];

  gCurves = Graphics3D[{saddlePoint[1], curve[1]},
    PlotRange → {{-X, X}, {-Y, Y}, {0, Z}}, Boxed → False, PlotLabel → lab,
    Axes → Automatic, AxesLabel → {Stl["x"], Stl["y"], Stl["F"]}}];

  lab = Stl@StringForm[
    ""\nSaddle points (dark green)\nand the steepest descent curve", "ek(z-Log[z])"];
  gSurface = Plot3D[Abs[F[x + I y, k]], {x, -X, X}, {y, -Y, Y},
    ImageSize → image, MeshFunctions → {#3 &}, Mesh → 10, Boxed → False,
    AxesLabel → {Stl["x"], Stl["y"], Stl["|f(z)|"]}, PlotLabel → lab];
  Show[{gSurface, gCurves}]

```

Out[24]=



From this it is clear that the approximate steepest descent curve is consistent with the requirements for the contour C .

4 Approximating the integral

For the purpose of calculating the leading asymptotic term, it is sufficient to approximate the steepest descent contour with $1 + iy$

Approximate $\rho[z]$ in the vicinity of the saddle point $z = 1$

In[25]:= `w4[1] = Normal[Series[$\rho[z]$, {z, 1, 2}]]`

Out[25]=
$$1 + \frac{1}{2} (-1 + z)^2$$

The integrand is then

In[26]:= `w4[2] = - $\frac{i k^{1-k}}{2 \pi}$ Exp[k ρ Approx] dz /. ρ Approx \rightarrow w4[1]`

Out[26]=
$$- \frac{i dz e^{k \left(1 + \frac{1}{2} (-1+z)^2\right)} k^{1-k}}{2 \pi}$$

Along the approximate contour

In[27]:= `w4[3] = w4[2] /. z \rightarrow (1 + I y) /. dz \rightarrow I dy`

Out[27]=
$$\frac{dy e^{k \left(1 - \frac{y^2}{2}\right)} k^{1-k}}{2 \pi}$$

In[28]:= `w4[4] = Integrate[w4[3], {y, - ∞ , ∞ }, Assumptions \rightarrow {Re[k] > 0}] /. dy \rightarrow 1`

Out[28]=
$$\frac{e^k k^{\frac{1}{2}-k}}{\sqrt{2 \pi}}$$

I check this against an expansion of $1/\Gamma(k)$.

In[29]:= `Normal@Series[1/Gamma[k], {k, ∞ , 0}] // ExpandAll // PowerExpand`

Out[29]=
$$\frac{e^k k^{\frac{1}{2}-k}}{\sqrt{2 \pi}}$$

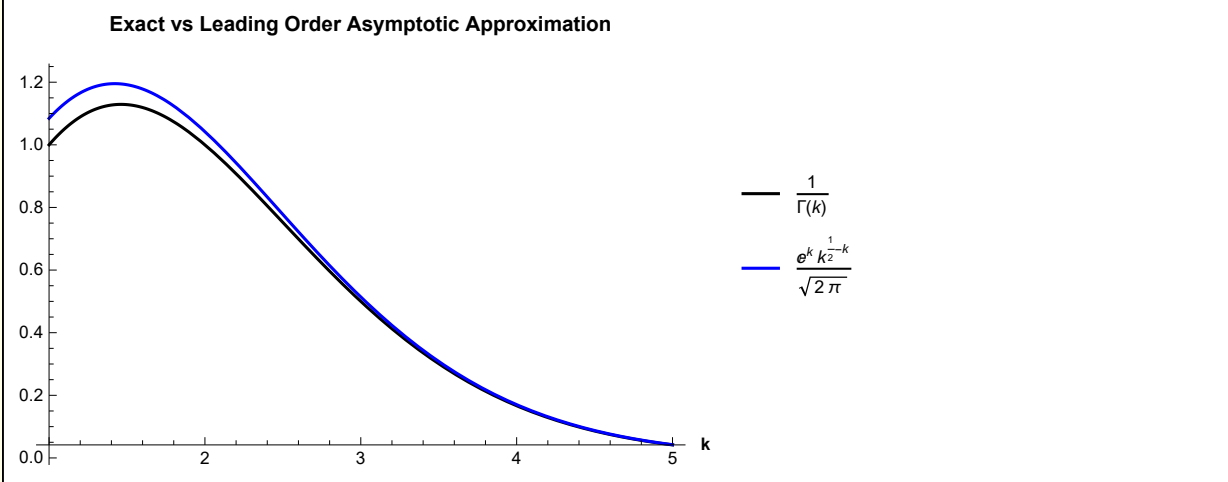
How good is the leading order approximation?

In[30]:=

```
Module[{lab, F},
  F[k_] :=  $\frac{e^k k^{\frac{1}{2}-k}}{\sqrt{2\pi}}$ ;

  lab = Stl@StringForm[
    "Exact vs Leading Order Asymptotic Approximation", TraditionalForm[F[x, k]]];
  Plot[{1/Gamma[k], F[k]}, {k, 1, 5}, PlotStyle -> {Black, Blue},
    PlotLegends -> Placed[{TraditionalForm[1/Gamma[k]], TraditionalForm[ $\frac{e^k k^{\frac{1}{2}-k}}{\sqrt{2\pi}}$ ]}, Right],
    AxesLabel -> {Stl["k"], Stl[""]}, PlotLabel -> lab, PlotRange -> All ]]
```

Out[30]=



Appendix

Programmatic simplification of

$$\frac{1}{2\pi i} \text{Exp}[z] z^{-k} dz$$

I simplified this expression by hand above. I now show how it could be accomplished programmatically. While not necessary in this case, one can imagine cases where the required manipulations could be quite complex.

In order to achieve the simplification above, it was necessary to use the identity $z^{-k} = e^{-k \text{Log}[z]}$. However, Mathematica automatically simplifies the rhs, viz.

In[31]:=

```
Exp[-k Log[z]]
z^-k
```

Out[31]=

To suppress this simplification, I temporarily change the name of the exponential term. Note that the

following expression simplifies automatically

In[32]:= **step[1] = w2[1] /. E → \mathcal{E} /. z^{-k} → $\mathcal{E}^{-k \text{Log}[z]}$**

Out[32]=
$$-\frac{i \, dz \, \mathcal{E}^{z-k \text{Log}[z]}}{2 \pi}$$

The next step is to expand the logarithm

In[33]:= **step[2] =
step[1] /. {z → k ζ , dz → k d ζ } /. Log[a_b_] → Log[a] + Log[b] // ExpandAll**

Out[33]=
$$-\frac{i \, d\zeta \, k \, \mathcal{E}^{k\zeta - k \text{Log}[k] - k \text{Log}[\zeta]}}{2 \pi}$$

I would like to further simplify the particular part of the argument of \mathcal{E} that is independent of ζ . This requires breaking the expression into parts and simplifying them independently. I write a function to accomplish this task.

In[37]:= **SpecialSimplify[step[2]]**

Out[37]=
$$-\frac{i \, d\zeta \, e^{k(\zeta - \text{Log}[\zeta])} \, k^{1-k}}{2 \pi}$$

In[35]:= **Clear[SpecialSimplify];
SpecialSimplify[a_ \mathcal{E}^b] :=
Module[{b ζ Independent, b ζ Dependent},
(* Separate b into parts independent and dependent on ζ *)
{b ζ Independent, b ζ Dependent} =
{Select[b, (FreeQ[#, ζ] &)], Select[b, (!FreeQ[#, ζ] &)]};
(* note that I reintroduce the exponential E and simplify the
 ζ independent parts and dependent parts separately *)
a Exp[b ζ Independent] Map[Factor, Exp[b ζ Dependent]]]**