BenderOrszag Ex10 p296 04-30-16

N. T. Gladd

Initialization: Be sure the files *NTGStylesheet2.nb* and *NTGUtilityFunctions.m* is are in the same directory as that from which this notebook was loaded. Then execute the cell immediately below by mousing left on the cell bar to the right of that cell and then typing “shift” + “enter”. Respond “Yes” in response to the query to evaluate initialization cells.

In[3]:=
```
SetDirectory[NotebookDirectory[]];
(* set directory where source files are located *)
SetOptions[EvaluationNotebook[], (* load the StyleSheet *)
 StyleDefinitions → Get["NTGStylesheet2.nb"]];
Get["NTGUtilityFunctions.m"]; (* Load utilities package *)
```

Original notebook — *BenderOrszag Ex10 p296 09-29-15*

Purpose

Approximation of a real integral using the saddle point method. In this example, there is an interesting choice of how to deform the original contour of integration to pass through the saddle point.

1 Example 10

For k → ∞, evaluate

$$\mathcal{I}(k) = \int_0^1 \mathrm{dx}\, e^{-4\,k\,z^2} \cos\left(5\,k\,z - k\,z^3\right)$$

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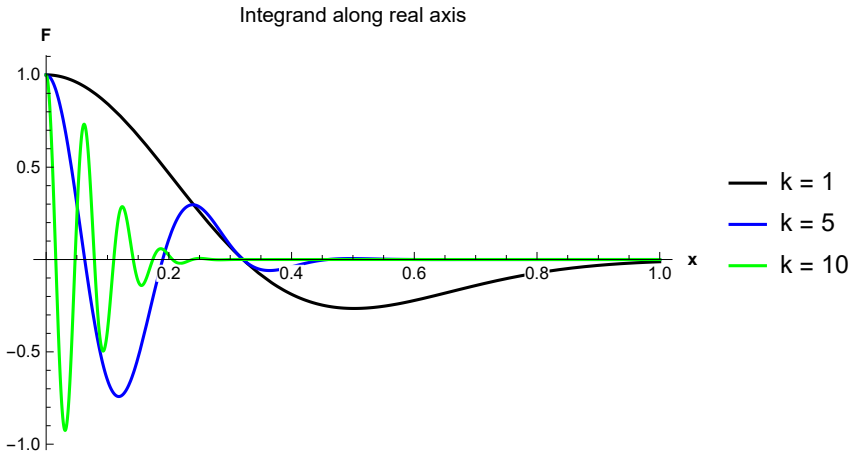
In[5]:=
```
Module[{lab, F},
 F[x_, k_] := Exp[-4 k x^2] Cos[5 k x - k x^3];

 lab = StringForm["Integrand along real axis", TraditionalForm[F[x, k]]];
 Plot[{F[x, 1], F[x, 5], F[x, 20]}, {x, 0, 1}, PlotStyle → {Black, Blue, Green},
  PlotLegends → Placed[{"k = 1", "k = 5", "k = 10"}, Right],
  AxesLabel → {Stl["x"], Stl["F"]}, PlotLabel → lab, PlotRange → All ]]
```

Out[5]=



Mathematica cannot evaluate this integral

In[6]:=
```
Integrate[Exp[-4 k x^2] Cos[5 k x - k x^3], {x, 0, 1}]
```

Out[6]=

$$\int_0^1 e^{-4 k x^2} \text{Cos}\left[5 k x - k x^3\right] \, dx$$

However, note that

In[7]:=
```
w1[1] = Exp[-4 k x^2] Cos[5 k x - k x^3] // TrigToExp
```

Out[7]=

$$\frac{1}{2} e^{5 i k x - 4 k x^2 - i k x^3} + \frac{1}{2} e^{-5 i k x - 4 k x^2 + i k x^3}$$

Then, by symmetry,

$$\mathcal{I}(k) = \frac{1}{2} \int_{-1}^{1} dx \, e^{5 i k x - 4 k x^2 - i k x^3}$$

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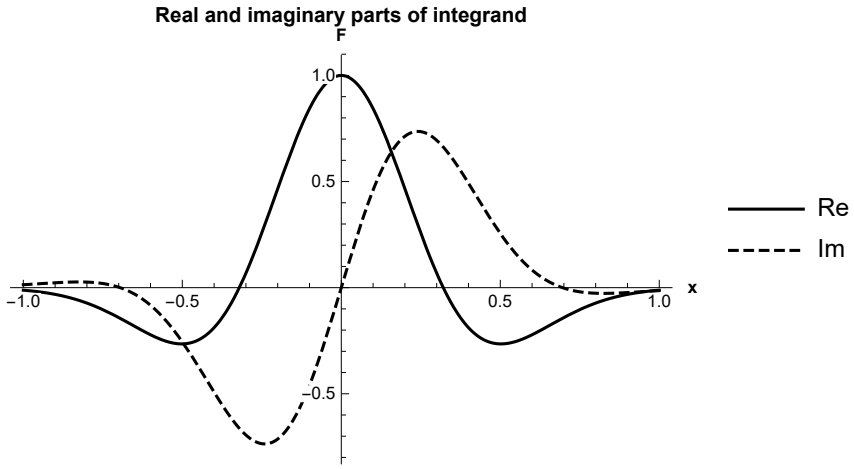
In[8]:=
```
Module[{k = 1, lab, F},
 F[x_, k_] := e^(5 i k x - 4 k x^2 - i k x^3);

 lab = Stl@
   StringForm["Real and imaginary parts of integrand", TraditionalForm[F[x, k]]];
 Plot[{Re[F[x, k]], Im[F[x, k]]}, {x, -1, 1}, PlotStyle →
   {Black, Directive[Black, Dashed]}, PlotLegends → Placed[{"Re", "Im"}, Right],
  AxesLabel → {Stl["x"], Stl["F"]}, PlotLabel → lab, PlotRange → All ]]
```

Out[8]=



Consistent with previous examples, I invoke the notation

$$\int \mathrm{dz}\, e^{k\rho(z)} \quad \text{where } \rho(z) = \phi(z) + i\psi(z)$$

In this case

In[9]:=
```
Clear[ρ];
ρ[z_] := 5 i z - 4 z^2 - i z^3
```

In[11]:=
```
w1[1] = ρ[z]
```

Out[11]=
$$5\,i\,z - 4\,z^2 - i\,z^3$$

The stationary points are two simple saddle points

In[12]:=
```
w1[2] = Solve[D[ρ[z], z] == 0]
```

Out[12]=
$$\left\{\{z \to i\}, \left\{z \to \frac{5\,i}{3}\right\}\right\}$$

The constant ψ curves ares

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In[13]:=
```
w1[3] = ϕ == ComplexExpand[Re[ρ[ x + I y] ]]
```
Out[13]=
$$\phi == -4\,x^2 - 5\,y + 3\,x^2\,y + 4\,y^2 - y^3$$

In[14]:=
```
w1[4] = ψ == ComplexExpand[Im[ρ[ x + I y] ]]
```
Out[14]=
$$\psi == 5\,x - x^3 - 8\,x\,y + 3\,x\,y^2$$

At the first saddle point z = i

In[15]:=
```
w1[7] = ρ[I]
```
Out[15]=
$$-2$$

The constant ψ curves passing through z = i are given by

In[16]:=
```
w1[8] = w1[4][[2]] == Im[ρ[I]]
```
Out[16]=
$$5\,x - x^3 - 8\,x\,y + 3\,x\,y^2 == 0$$

In[17]:=
```
w1[9] = Solve[w1[8], x]
```
Out[17]=
$$\left\{\{x \to 0\}, \left\{x \to -\sqrt{5 - 8\,y + 3\,y^2}\right\}, \left\{x \to \sqrt{5 - 8\,y + 3\,y^2}\right\}\right\}$$

Similarly, at the second saddle point z = 5 i/3

In[18]:=
```
w1[10] = ρ[5 I / 3]
```
Out[18]=
$$-\frac{50}{27}$$

The constant ψ curves passing through z =5 i/3 are given by

In[19]:=
```
w1[8] = w1[4][[2]] == Im[ρ[5 I / 3]]
```
Out[19]=
$$5\,x - x^3 - 8\,x\,y + 3\,x\,y^2 == 0$$

In[20]:=
```
Solve[w1[8], y]
```
Out[20]=
$$\left\{\left\{y \to \frac{1}{3}\left(4 - \sqrt{1 + 3\,x^2}\right)\right\}, \left\{y \to \frac{1}{3}\left(4 + \sqrt{1 + 3\,x^2}\right)\right\}\right\}$$

In[21]:=
```
w1[9] = Solve[w1[8], x]
```
Out[21]=
$$\left\{\{x \to 0\}, \left\{x \to -\sqrt{5 - 8\,y + 3\,y^2}\right\}, \left\{x \to \sqrt{5 - 8\,y + 3\,y^2}\right\}\right\}$$

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In[22]:=

```
Module[{k = 2, X = 2, Y = 2, Z = 0.2, δF = 0.02, image = 400, lab,
  F, saddlePoint, curve, CurveXofY, SaddlePoint, gSurface, gCurves},
 F[z_, k_] := Exp[k ρ[z]];
 CurveXofY[fXofY_, {yStart_, yFin_, dy_}] :=
  {GREEN, Tooltip[Line@Table[{fXofY[y], y, Abs@F[fXofY[y] + I y , k] + δF},
      {y, yStart, yFin, dy}], StringForm["x = ``", TraditionalForm[fXofY[y]]]]};
 SaddlePoint[zSP_] := {DARKGREEN, PointSize[0.03], Tooltip[
    Point[{Re[zSP] , Im[zSP], Abs@F[zSP, k] + δF}], StringForm["z_SP = ``", zSP]]};

 saddlePoint[1] = SaddlePoint[I];
 saddlePoint[2] = SaddlePoint[5 I / 3];
 curve[1] = CurveXofY[(0) &, {-Y, 2 Y, 0.05}];
 curve[2] = CurveXofY[Function[{y}, -Sqrt[5 - 8 y + 3 y^2] ], {-Y, 1, 0.05}];
 curve[3] = CurveXofY[Function[{y}, Sqrt[5 - 8 y + 3 y^2] ], {-Y, 1, 0.05}];
 curve[4] = CurveXofY[Function[{y}, -Sqrt[5 - 8 y + 3 y^2] ], {5 / 3, 3, 0.05}];
 curve[5] = CurveXofY[Function[{y}, Sqrt[5 - 8 y + 3 y^2] ], {5 / 3, 3, 0.05}];
 gCurves = Graphics3D[{saddlePoint[1], saddlePoint[2], curve[1], curve[2], curve[3],
    curve[4], curve[5]}, PlotRange → {{-X, X}, {-Y, Y}, {0, Z}}, Boxed → False,
   PlotLabel → lab, Axes → Automatic, AxesLabel → {Stl["x"], Stl["y"], Stl["F"]}];

 lab = Stl@StringForm[
    "``\nThe two saddle points (dark green)\n and the steepest ascent/descent
      curves(green)", e^(5 i k z - 4 k z^2 - i k z^3)];
 gSurface = Plot3D[Abs[F[x + I y, k]], {x, -X, X}, {y, -Y / 2, 3 Y / 2},
   ImageSize → image, MeshFunctions -> {#3 &}, Mesh → 10,
   Boxed → False, AxesLabel → {Stl["x"], Stl["y"], Stl["|f(z)|"]},
   PlotRange → {{-X, X}, {-Y / 2, 3 Y / 2}, {0, Z}}, PlotLabel → lab];
 Show[{gSurface, gCurves}]]
```

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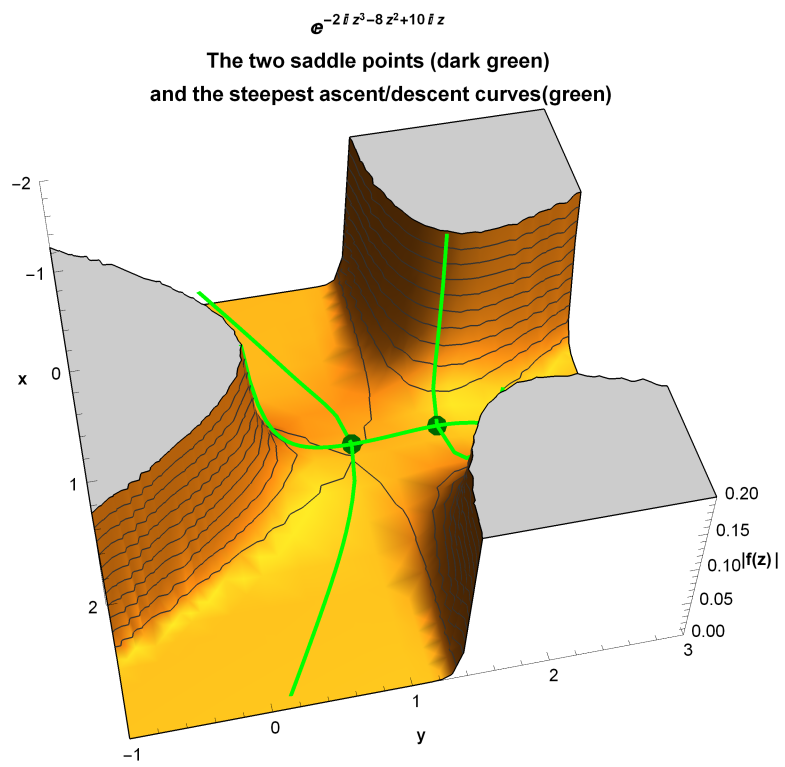
Out[22]=



The integral is approximated by deforming the contour of integration to pass through the saddle point at z = i. For that purpose I also need to calculate the constant ψ curves passing through the integration end points

In[23]:=
```
w1[11] = ρ[-1]
```

Out[23]= $-4-4\,i$

In[24]:=
```
w1[12] = 5 x - x^3 - 8 x y + 3 x y^2 == Im[ρ[-1]]
```

Out[24]= $5\,x - x^3 - 8\,x\,y + 3\,x\,y^2 == -4$

In[25]:=
```
w1[13] = Solve[w1[12], y]
```

Out[25]= $\left\{\left\{y \to \frac{4\,x - \sqrt{-12\,x + x^2 + 3\,x^4}}{3\,x}\right\}, \left\{y \to \frac{4\,x + \sqrt{-12\,x + x^2 + 3\,x^4}}{3\,x}\right\}\right\}$

In[26]:=
```
w1[13] /. x → -1
```

Out[26]= $\left\{\left\{y \to \frac{8}{3}\right\}, \{y \to 0\}\right\}$

So the second root corresponds to a constant ψ curve that passes through z = {-1, 0}.

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In[27]:=
```
w1[14] = ρ[1]
```

Out[27]=
$$-4+4\,i$$

In[28]:=
```
w1[15] = 5 x - x^3 - 8 x y + 3 x y^2 == Im[ρ[1]]
```

Out[28]=
$$5\,x - x^3 - 8\,x\,y + 3\,x\,y^2 == 4$$

In[29]:=
```
w1[16] = Solve[w1[15], y]
```

Out[29]=
$$\left\{\left\{y \to \frac{4\,x - \sqrt{12\,x + x^2 + 3\,x^4}}{3\,x}\right\}, \left\{y \to \frac{4\,x + \sqrt{12\,x + x^2 + 3\,x^4}}{3\,x}\right\}\right\}$$

In[30]:=
```
w1[16] /. x → 1
```

Out[30]=
$$\left\{\{y \to 0\}, \left\{y \to \frac{8}{3}\right\}\right\}$$

So the first root corresponds to a constant ψ curve that passes through z = {1, 0}.

Note: if viewing with Mathematica, mousing over the curves and saddle points accesses Tooltip info.

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In[31]:=

```
Module[{k = 2, X = 2, Y = 2, Z = 1, δF = 0.02, image = 400, lab, F, saddlePoint, curve,
  integrationPath , CurveXofY, SaddlePoint, CurveYofX, gSurface, gCurves},
 F[z_, k_] := Exp[k ρ[z]];
 CurveXofY[fXofY_, {yStart_, yFin_, dy_}] :=
  {GREEN, Tooltip[Line@Table[{fXofY[y], y, Abs@F[fXofY[y] + I y , k] + δF},
      {y, yStart, yFin, dy}], StringForm["x = ``", TraditionalForm[fXofY[y]]]]};
 SaddlePoint[zSP_] := {DARKGREEN, PointSize[0.03], Tooltip[
    Point[{Re[zSP] , Im[zSP], Abs@F[zSP, k] + δF}], StringForm["z_SP = ``", zSP]]};
 CurveYofX[fYofX_, {xStart_, xFin_, dx_}] := {BLUE, Tooltip[
    Line@Table[{x, fYofX[x], Abs@F[x + I fYofX[x], k] + δF}, {x, xStart, xFin, dx}],
    StringForm["y = ``", TraditionalForm[fYofX[x]]]]};

 saddlePoint[1] = SaddlePoint[I];
 saddlePoint[2] = SaddlePoint[5 I / 3];
 curve[1] = CurveXofY[(0) &, {-Y, 2 Y, 0.05}];
 curve[2] = CurveXofY[Function[{y}, -Sqrt[5 - 8 y + 3 y^2] ], {-Y, 1, 0.05}];
 curve[3] = CurveXofY[Function[{y}, Sqrt[5 - 8 y + 3 y^2] ], {-Y, 1, 0.05}];
 curve[4] = CurveXofY[Function[{y}, -Sqrt[5 - 8 y + 3 y^2] ], {5 / 3, 3, 0.05}];
 curve[5] = CurveXofY[Function[{y}, Sqrt[5 - 8 y + 3 y^2] ], {5 / 3, 3, 0.05}];

 integrationPath = {RED, PointSize[0.03],
   Point[{-1, 0, Abs[F[-1, k]] + δF }], Point[{1, 0, Abs[F[1, k]] + δF }],
   Line[Table[{x, 0, Abs[F[x, k]] + δF }, {x, -1, 1, 0.05}]]};

 curve["Minus One"] =
  CurveYofX[Function[{x}, (4 x + Sqrt[-12 x + x^2 + 3 x^4]) / (3 x)], {-X, -0.5, 0.05}];
 curve["Plus One"] = CurveYofX[Function[{x}, (4 x - Sqrt[12 x + x^2 + 3 x^4]) / (3 x)], {0.5, X, 0.05}];

 gCurves = Graphics3D[{saddlePoint[1], saddlePoint[2], curve[1], curve[2], curve[3],
    curve[4], curve[5], integrationPath, curve["Minus One"], curve["Plus One"]},
   PlotRange → {{-X, X}, {-Y, Y}, {0, Z}}, Boxed → False, PlotLabel → lab,
   Axes → Automatic, AxesLabel → {Stl["x"], Stl["y"], Stl["F"]}];

 lab = Stl@StringForm[
    "``\nThe two saddle points (dark green)\n and the steepest ascent/descent
      curves(green)\nconstant ψ curves through
      integration endpoints (blue)", e^(5 i k z - 4 k z^2 - i k z^3)];
 gSurface = Plot3D[Abs[F[x + I y, k]], {x, -X, X}, {y, -Y / 2, 3 Y / 2},
   ImageSize → image, MeshFunctions -> {#3 &}, Mesh → 10,
   Boxed → False, AxesLabel → {Stl["x"], Stl["y"], Stl["|f(z)|"]},
   PlotRange → {{-X, X}, {-Y / 2, 3 Y / 2}, {0, Z}}, PlotLabel → lab];
 Show[{gSurface, gCurves}]]
```

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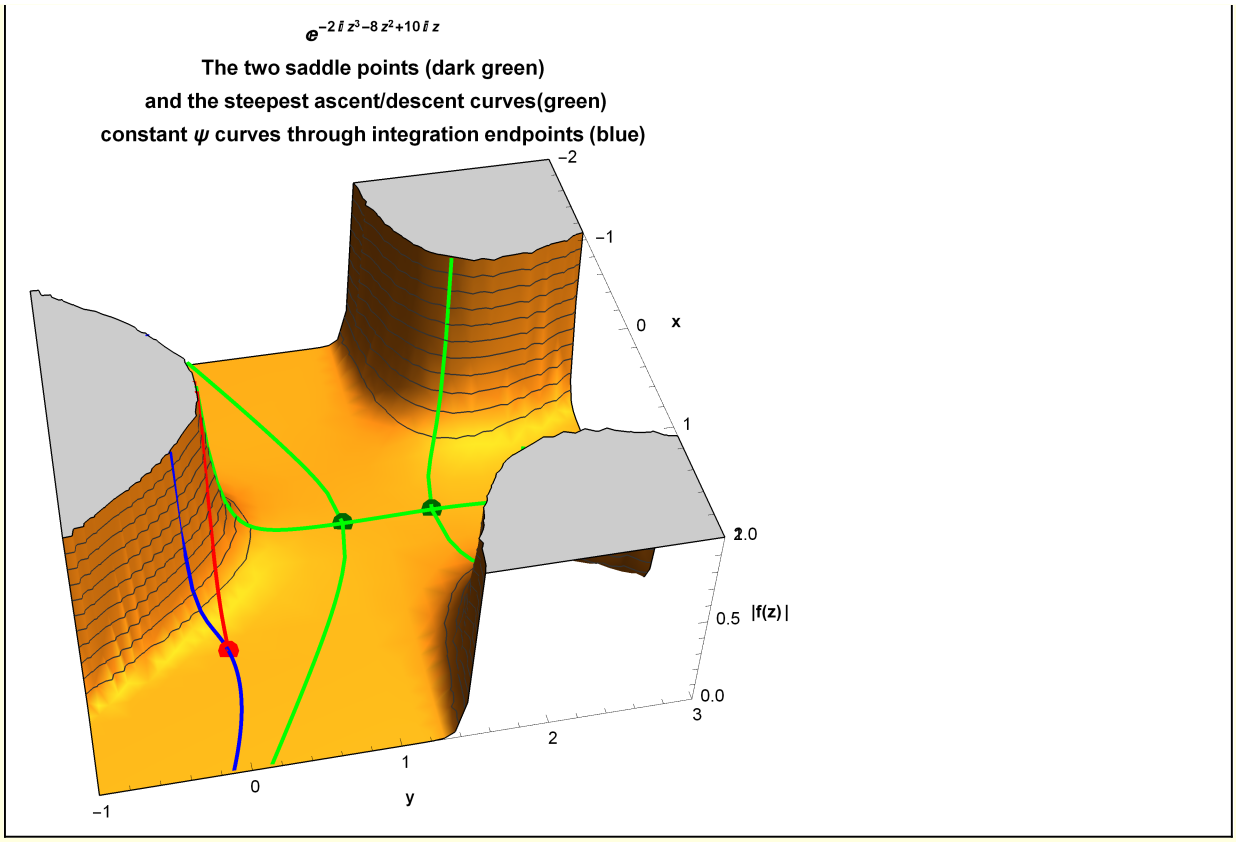
Out[31]=



2 Example 10 - approximating the integral

The integral is approximated by deforming the original contour of integral — along the real axis from x = -1 to x = +1 — to pass through the saddle point at z = i. The deformation is accomplished by choosing a contour C_1 that leaves the integration end point z = -1 and follows a steepest descent curve to -∞ (the blue curve that descends from point z = -1. That curve is joined to the steepest descent curve that runs from the saddle point at z = i out to z = -∞ (the contour C_2). The contribution from the contour connecting C_1 and C_2 at ∞ is negligibly small compared to the contribution near the saddle point. Similarly, the contour C_2 is extended to z = ∞ and joined with the contour C_3 that connects with the integration endpoint at z = 1.

In this example, this deformed contour is quite different from the original contour.

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In[32]:=

```
Module[{X = 3, Y = 2, saddlePoints,
  integrationPath, C2Approximate, curve, lab, CurveXofY, CurveYofX},
 CurveXofY[f_, {yStart_, yFin_, dy_}, {label_, yPos_}] :=
  {Arrowheads[{0.0, 0.05, 0.05, 0.0}], Arrow@Table[{f[y], y}, {y, yStart, yFin, dy}],
   {Black, Stl@Text[label, {f[yPos], yPos}]}};
 CurveYofX[f_, {xStart_, xFin_, dx_}, {label_, xPos_}] :=
  {Arrowheads[{0.0, 0.05, 0.05, 0.0}], Arrow@Table[{x, f[x]}, {x, xStart, xFin, dx}],
   {Black, Stl@Text[label, {xPos, f[xPos]}]}};
 saddlePoints = {PointSize[0.03], {Green, Point[{0, 1}], Point[{{0, 5/3}}]},
   {Red, Point[{{-1, 0}}], Point[{{1, 0}}]}};
 integrationPath = {RED, Arrowheads[{0.0, 0.05, 0.05, 0.0}],
   Arrow[{{-1, 0}, {1, 0}}], {Black, Stl@Text[" C_original ", {0, 0}]}};
 C2Approximate = {Directive[Green, Dashed], Arrowheads[{0.0, 0.05, 0.05, 0.0}],
   Arrow[{{-X, 1}, {X, 1}}], {Black, Stl@Text[" C_2, approximate ", {1.5, 1}]}};
 curve[1] = CurveXofY[Function[{y}, -Sqrt[5 - 8 y + 3 y^2]], {-Y, 1, 0.05}, {" C_2 ", 0}];
 curve[2] = CurveXofY[Function[{y}, Sqrt[5 - 8 y + 3 y^2]], {1, -Y, -0.05}, {" C_2 ", 0}];
 curve[3] =
  CurveYofX[Function[{x}, (4 x + Sqrt[-12 x + x^2 + 3 x^4])/(3 x)], {-1, -X, -0.05}, {" C_1 ", -2.5}];
 curve[4] = CurveYofX[Function[{x}, (4 x - Sqrt[12 x + x^2 + 3 x^4])/(3 x)],
   {X, 1, -0.05}, {" C_3 ", 2.5}];
 lab = Stl["Original and Deformed Contour"];
 Graphics[{{saddlePoints, integrationPath, C2Approximate},
   {Directive[Green, Thick], curve[1], curve[2]},
   {Directive[Blue, Thick], curve[3], curve[4]}}, Axes → True,
  AspectRatio → 1, PlotRange → {{-X, X}, {-(Y - 1), (Y)}},
  PlotLabel → lab, AxesLabel → {Stl["x"], Stl["y"]}]]
```

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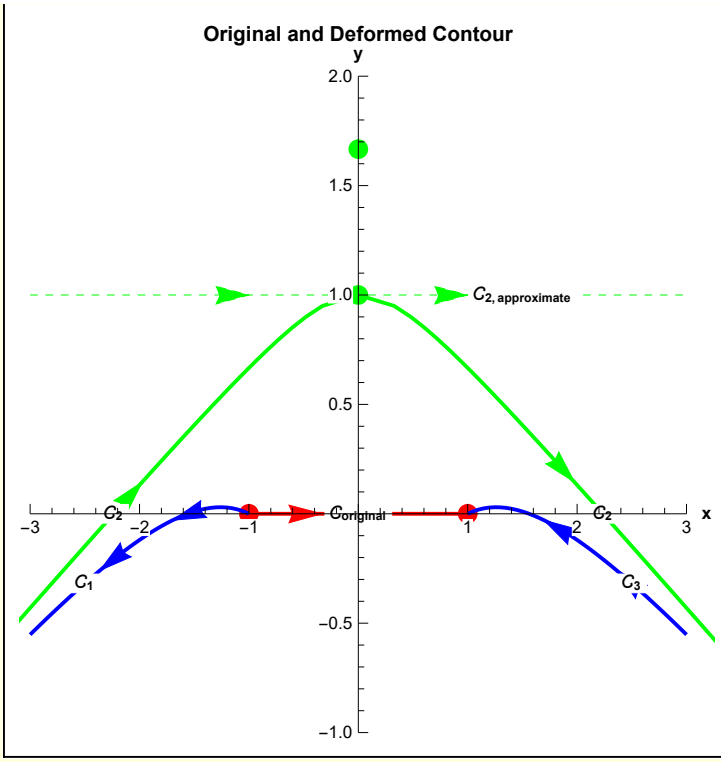
Out[32]=

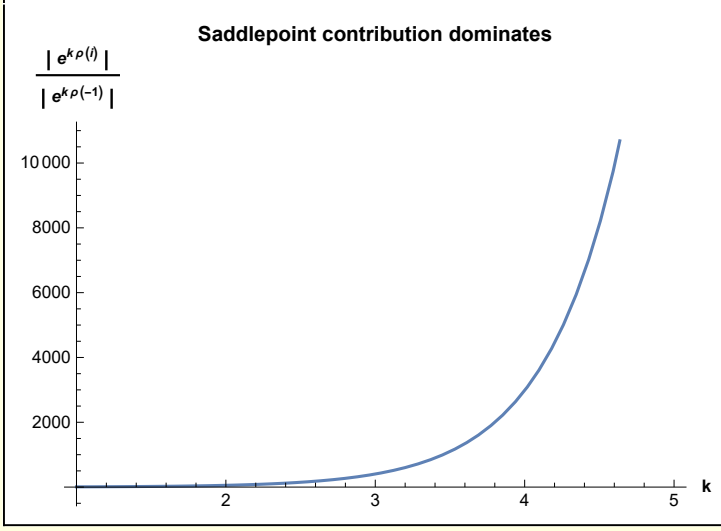


The contribution from C_2 dominates.

In[33]:=

```
Plot[{Abs@Exp[k ρ[I]] / Abs@Exp[k ρ[-1]]}, {k, 1, 5},
 PlotLabel → Stl["Saddlepoint contribution dominates"],
 AxesLabel → {Stl["k"], Stl["|e^(k ρ (i))| / |e^(k ρ (-1))|"]}]
```

Out[33]=



For the purpose of determining the leading order asymptotic contribution, it is sufficient to approximate the contour C_2 by a straight line.

Approximate ρ[z] in the vicinity of the saddle point z = i

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In[34]:=
```
w2[1] = Normal[Series[ρ[z], {z, I, 2}]]
```

Out[34]=
$$-2-\left(-i+z\right)^2$$

In[35]:=
```
w2[2] = 1/2 Exp[k ρApprox] dz /. ρApprox → w2[1]
```

Out[35]=
$$\frac{1}{2}\,dz\,e^{k\left(-2-(-i+z)^2\right)}$$

Along the approximate contour

In[36]:=
```
w2[3] = w2[2] /. z → (u + I) /. dz → du
```

Out[36]=
$$\frac{1}{2}\,du\,e^{k\left(-2-u^2\right)}$$

In[37]:=
```
w2[4] = Integrate[w2[3], {u, -∞, ∞}, Assumptions → {Re[k] > 0}] /. du → 1
```

Out[37]=
$$\frac{e^{-2k}\sqrt{\pi}}{2\sqrt{k}}$$

I compare this approximation against a numerical calculation

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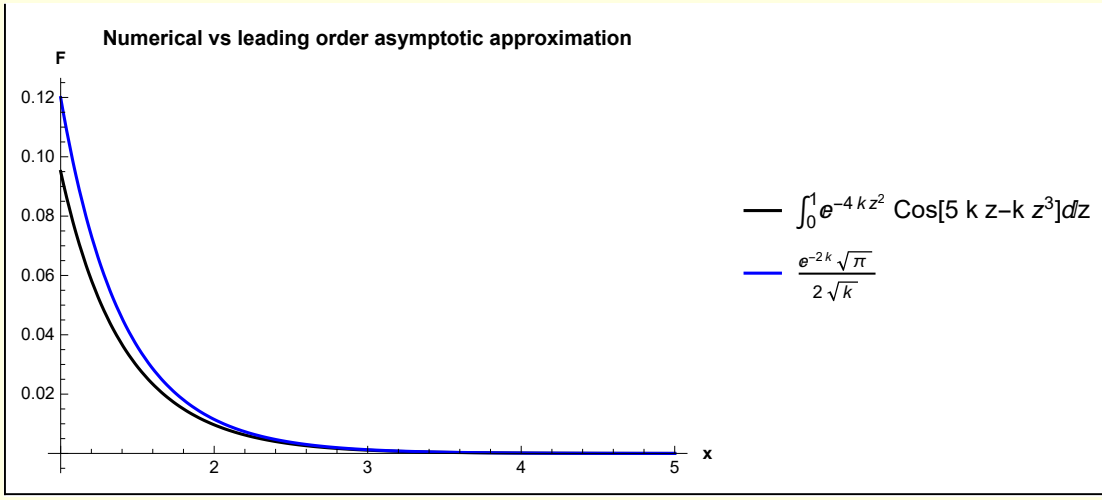
In[38]:=
```
Module[{lab, F, FAsymptotic},
 F[k_] := NIntegrate[Exp[-4 k z^2] Cos[5 k z - k z^3], {z, 0, 1}];
 FAsymptotic[k_] := e^(-2 k) Sqrt[π] / (2 Sqrt[k]);

 lab = Stl@StringForm["Numerical vs leading order asymptotic approximation"];
 Plot[{F[k], FAsymptotic[k]}, {k, 1, 5}, PlotStyle → {Black, Blue},
  PlotLegends → Placed[{"∫_0^1 e^(-4 k z^2) Cos[5 k z-k z^3]dz", "e^(-2 k) Sqrt[π] / (2 Sqrt[k])"}, Right],
  AxesLabel → {Stl["x"], Stl["F"]}, PlotLabel → lab, PlotRange → All ]]
```

NIntegrate: The integrand $e^{-4 k z^2} \text{Cos}[5 k z - k z^3]$ has evaluated to non-numerical values for all sampling points in the region with boundaries {{0, 1}}.

NIntegrate: The integrand $e^{-4 k z^2} \text{Cos}[5 k z - k z^3]$ has evaluated to non-numerical values for all sampling points in the region with boundaries {{0, 1}}.

Out[38]=



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