BenderOrszag Ex11 p298 07-18-16

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Initialization: Be sure the files *NTGStylesheet2.nb* and *NTGUtilityFunctions.m* is are in the same directory as that from which this notebook was loaded. Then execute the cell immediately below by mousing left on the cell bar to the right of that cell and then typing "shift" + "enter". Respond "Yes" in response to the query to evaluate initialization cells.

In[3]:=
```
SetDirectory[NotebookDirectory[]];
(* set directory where source files are located *)
SetOptions[EvaluationNotebook[], (* load the StyleSheet *)
 StyleDefinitions → Get["NTGStylesheet2.nb"]];
Get["NTGUtilityFunctions.m"]; (* Load utilities package *)
```

Original notebook — *BenderOrszag Ex11 p298 10-02-15*

Purpose

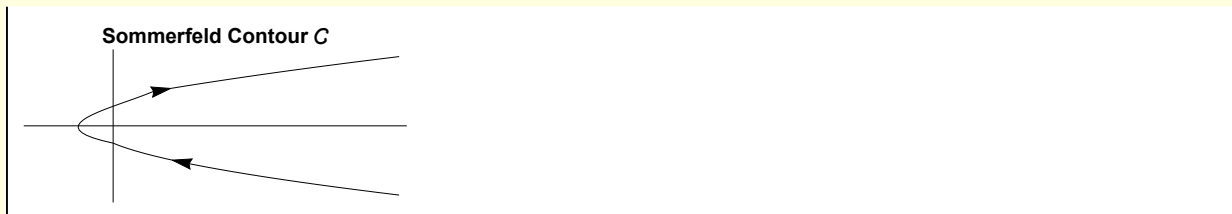The problem is to determine the leading order asymptotic expansion of the Bessel function

$$J_k(k) \ \text{as } k \ \to \ \infty$$

1 Example 11, p 298

The Bessel function $J_v(x)$ has the integral representation

$$J_v(k) \ = \ \frac{1}{2\pi i} \int_C dz \, e^{k \sinh(z) - v z}$$

where C is the Sommerfeld contour $\infty - i\pi$ to $\infty + i\pi$

Sommerfeld Contour C

For the problem at hand, take $v = k$

$$J_v(k) \ = \ \frac{1}{2\pi i} \int_C dz \, e^{k(\sinh(z) - z)}$$
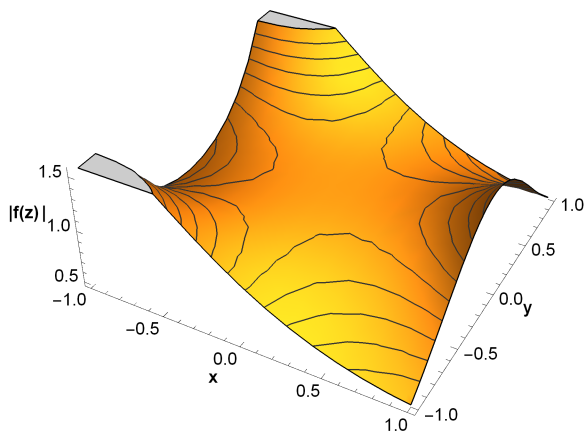
2 Calculation

Visualize the integrand

In[5]:=
```
Module[{k = 2, X = 1, Y = 1, Z = 2,
   δF = 0.005, image = 300, gSurface, lab, F, ψConstant},
  F[z_, k_] := Exp[ k (Sinh[z] - z)];

  gSurface = Plot3D[Abs[F[x + I y, k]], {x, -X, X},
    {y, -Y, Y}, ImageSize → image, MeshFunctions -> {#3 &}, Mesh → 10,
    Boxed → False, AxesLabel → {Stl["x"], Stl["y"], Stl["|f(z)|"]}];
  Show[{gSurface}]]
```

Out[5]=

The standard form I have been using for steepest descent problems is

$$\int dz\, e^{k\,\rho(z)} \;=\; \int dz\, e^{k\,(\phi(z)+i\,\psi(z))}$$

But, in this case the integrand as defined is

In[6]:=
```
w2[1]  =   1/(2 π I) Exp[ k (Sinh[z] - z)] dz
```

Out[6]=
$$-\,\frac{\mathbb{i}\; dz\; e^{k\,(-z+\text{Sinh}[z])}}{2\,\pi}$$

and so ρ is given by

In[7]:=
```
w2[2]  = ρ[z] == (w2[1] /. a_ Exp[k ρ_] → ρ)
```

Out[7]=
$$\rho[z] \;==\; -z + \text{Sinh}[z]$$

In[8]:=
```
w2[3] = D[#, z] & /@ w2[2]
```

Out[8]=
$$\rho'[z] == -1 + \text{Cosh}[z]$$

The saddle point is

In[10]:=
```
w2[4] = Solve[(w2[3] /. ρ'[z] → 0), z][[1, 1]] /. C[1] → 0
```

Out[10]=
$$z \to 0$$

Note that this is a higher order saddle point $\rho'' = 0$ and $\rho''' \neq 0$

In[11]:=
```
w2[5] = {D[w2[2], {z, 2}], D[w2[2], {z, 3}]} /. w2[4]
```

Out[11]=
$$\left\{\rho''[0] == 0, \rho^{(3)}[0] == 1\right\}$$

In[12]:=
```
w2[6] = w2[2][[2]] /. z → x + I y // ComplexExpand
```

Out[12]=
$$-x + \dot{\mathbb{i}}\left(-y + \text{Cosh}[x]\,\text{Sin}[y]\right) + \text{Cos}[y]\,\text{Sinh}[x]$$

The expression for ψ is

In[13]:=
```
w2[7] = -y + Cosh[x] Sin[y]
```
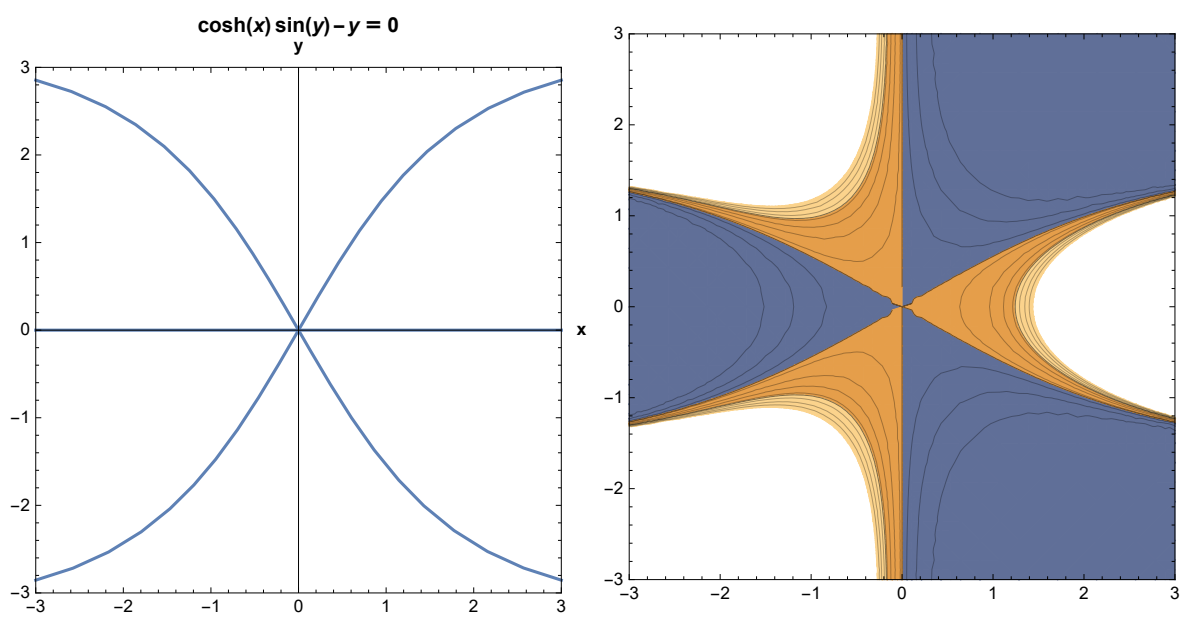
Out[13]=
$$-y + \text{Cosh}[x]\,\text{Sin}[y]$$

The steepest descent and assent curves are given by $\psi = 0$ and can be visualized using RegionPlot.

```
Module[{X = 3, Y = 3, image = 300, ℛ, g},
  ℛ = ImplicitRegion[-y + Cosh[x] Sin[y] == 0, {x, y}];
  g[1] = RegionPlot[ℛ, PlotRange → {{-X, X}, {-Y, Y}},
     Axes → True, ImageSize → image, AxesLabel → {Stl["x"], Stl["y"]},
     PlotLabel → Stl[TraditionalForm[-y + Cosh[x] Sin[y] == 0]]];
  g[2] =
   Module[{k = 2, Z = 3, gSurface, lab, F, ψConstant},
     F[z_, k_] := Exp[k (Sinh[z] - z)];

     gSurface =
      ContourPlot[Abs[F[x + I y, k]], {x, -X, X}, {y, -Y, Y}, ImageSize → image,
       MeshFunctions -> {#3 &}, Mesh → 10, PlotRange → {{-X, X}, {-Y, Y}, {-Z, Z}}]];
  Grid[{{g[1], g[2]}}]]
```


The steepest descent curves of are 1) along the negative x axis, 2) the curve in the first quadrant, and 2) the curve in the fourth quadrant.

I seek approximate expressions for these curves. First, consider y << 1

In[15]:= `w2[8] = Normal@Series[w2[7], {y, 0, 3}]`

Out[15]= $y (-1 + Cosh[x]) - \frac{1}{6} y^3 Cosh[x]$

In[16]:= `w2[9] = Solve[w2[8] == 0, y]`

Out[16]= $\{\{y \to 0\}, \{y \to -\sqrt{6} \sqrt{1 - Sech[x]}\}, \{y \to \sqrt{6} \sqrt{1 - Sech[x]}\}\}$

Now consider small x

In[17]:= `w2[10] = y == Normal@Series[`$\sqrt{6}$ $\sqrt{1 - \text{Sech}[x]}$`, {x, 0, 2}]`

Out[17]= `y ==` $\sqrt{3}$ `x`

what angle does this correspond to?

In[18]:= `ArcTan[`$\sqrt{3}$`]`

Out[18]= $\dfrac{\pi}{3}$

The Sommerfeld contour can be distorted to pass through the saddle point z = 0 along the steepest descent curves C_1 and C_2.

In[19]:=

```
Module[{k , X = 2, Y = 3, Z = 1.1, δF = 0.01, image = 300, xC1, sdCurve1,
   sdCurve2, sdCurveApprox1, sdCurveApprox2, saddlePoint, gSurface, lab, F},
  F[z_, k_] := e^(k (Sinh[z] - z));
  xC1[y_] := ArcCosh[y Csc[y]];

  sdCurve1 = {Directive[Yellow, Thick], Arrowheads[{0.05, 0.05, 0.05, 0.05}], Arrow@
     Table[{xC1[y], y}, {y, 0.1, Y, 0.1}], {Black, Text[Stl["C₂"], {xC1[2], 2 }]}};
  sdCurveApprox1 = {Directive[Yellow, Thick, Dashed],
    Line@Table[{x, √3 x}, {x, 0.1, X, 0.1}]};
  sdCurve2 = {Directive[Yellow, Thick], Arrowheads[{0.05, 0.05, 0.05, 0.05}],
    Arrow@Table[{xC1[y], y}, {y, -Y, -0.1, 0.1}]};
  sdCurveApprox2 = {Directive[Yellow, Thick, Dashed],
    Line@Table[{x, -√3 x}, {x, 0.1, X, 0.1}],
    {Black, Text[Stl["C₁"], {xC1[-2], -2 }]}};

  saddlePoint = {Yellow, PointSize[0.025], Point[{0, 0}]};

  lab = Stl[Stl[TraditionalForm[F[z, k]]]];
  k = 2;
  gSurface =
   ContourPlot[Abs[F[x + I y, k]], {x, -X, X}, {y, -Y, Y}, ImageSize → image,
    MeshFunctions -> {#3 &}, Mesh → 50 , PlotRange → {{-X, X}, {-Y, Y}, {0, Z}},
    Epilog → {saddlePoint, sdCurve1, sdCurve2, sdCurveApprox1, sdCurveApprox2},
    FrameLabel → {{ Stl[Rotate["y", -π/2]], ""}, {Stl["x"], lab }}]]
```
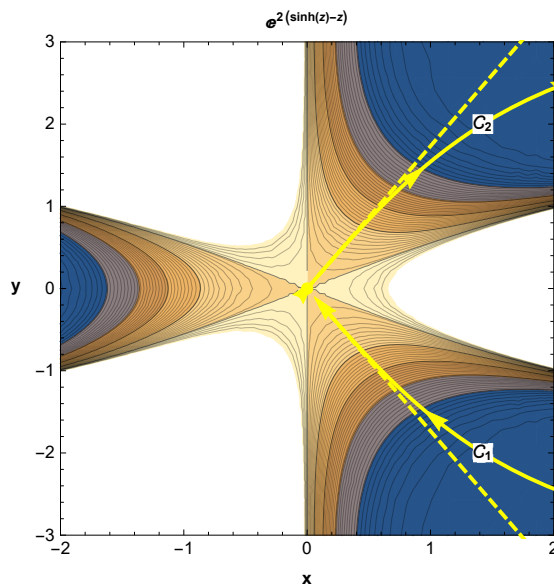
Out[19]=

3 Leading order asymptotic expansion

The steepest descent approach to the integral is

$$J_\nu(k) = \frac{1}{2\pi i} \int_C dz\, e^{k\sinh(z) - kz}$$

$$= \frac{1}{2\pi i} \int_{C_1} dz\, e^{k(\sinh(z) - z)} + \frac{1}{2\pi i} \int_{C_2} dz\, e^{k(\sinh(z) - z)}$$

For the purpose of calculating the leading order asymptotic approximation, C_1 will be approximated by $e^{-i\pi/3}$ and C_2 by $e^{i\pi/3}$.

Near the saddle point z = 0, $\rho(z)$ is approximated by

In[20]:=
```
w3[1] = ρ[z] → Normal@Series[Sinh[z] - z, {z, 0, 3}]
```

Out[20]=
$$\rho[z] \to \frac{z^3}{6}$$

so the integrand is

In[21]:=
```
w3[2] = 1/(2 π I) dz Exp[k ρ[z]] /. w3[1]
```

Out[21]=
$$-\frac{i\, dz\, e^{\frac{k z^3}{6}}}{2\pi}$$

Along C_1

In[22]:=
```
w3[3] = w3[2] /. z → r Exp[-I π/3] /. dz → dr Exp[-I π/3]
```

Out[22]=
$$-\frac{i\, dr\, e^{-\frac{i\pi}{3} - \frac{k r^3}{6}}}{2\pi}$$

Along C_2

In[23]:=
```
w3[4] = w3[2] /. z → r Exp[I π/3] /. dz → dr Exp[I π/3]
```

Out[23]=
$$-\frac{i\, dr\, e^{\frac{i\pi}{3} - \frac{k r^3}{6}}}{2\pi}$$

Combining

In[24]:=
```
w3[5] = Int[w3[3], {r, -∞, 0}] + Int[w3[4], {r, 0, ∞}]
```

Out[24]=
$$\text{Int}\left[-\frac{i\, dr\, e^{-\frac{i\pi}{3} - \frac{k r^3}{6}}}{2\pi}, \{r, -\infty, 0\}\right] + \text{Int}\left[-\frac{i\, dr\, e^{\frac{i\pi}{3} - \frac{k r^3}{6}}}{2\pi}, \{r, 0, \infty\}\right]$$

Reverse the order of integration in the first term

In[25]:=
```
w3[6] = w3[5] /. Int[a_, {r, -∞, 0}] → -Int[a, {r, 0, ∞}]
```

Out[25]=
$$-\text{Int}\left[-\frac{\mathbb{i}\, dr\, e^{-\frac{i\pi}{3}-\frac{k r^3}{6}}}{2\pi}, \{r, 0, \infty\}\right] + \text{Int}\left[-\frac{\mathbb{i}\, dr\, e^{\frac{i\pi}{3}-\frac{k r^3}{6}}}{2\pi}, \{r, 0, \infty\}\right]$$

In[26]:=
```
w3[7] = w3[6] /. Int → Integrate /. dr → 1
```

Out[26]=
$$\text{ConditionalExpression}\left[\frac{(-1)^{1/6}\, 3^{1/3}\, \text{Gamma}\left[\frac{4}{3}\right]}{2^{2/3}\, k^{1/3}\, \pi} - \frac{(-1)^{5/6}\, 3^{1/3}\, \text{Gamma}\left[\frac{4}{3}\right]}{2^{2/3}\, k^{1/3}\, \pi}, \text{Re}[k] > 0\right]$$

and the leading order asymptotic expansion is

In[27]:=
```
w3[8] = Simplify[w3[7], Re[k] > 0]
```

Out[27]=
$$\frac{3^{5/6}\, \text{Gamma}\left[\frac{4}{3}\right]}{2^{2/3}\, k^{1/3}\, \pi}$$

I check against the Mathematica evaluation of BesselJ[k, k]

In[28]:=
```
Module[{lab, standardArgs, F, FAsymptotic},
  FAsymptotic[k_] := (3^{5/6} Gamma[4/3])/(2^{2/3} k^{1/3} π);

  lab = Stl@StringForm["Mathematica vs steepest descent approximation"];
  Plot[{BesselJ[k, k], FAsymptotic[k]}, {k, 0.5, 5}, AxesLabel → {Stl["k"], Stl["F"]},
   PlotLabel → lab, PlotStyle → {Black, Blue, Darker[Green, 0.5]},
   PlotLegends → Placed[{"BesselJ[k,k]", "(3^{5/6} Gamma[4/3])/(2^{2/3} k^{1/3} π)"}, Right]]]
```

Out[28]=