Jackson Electrostatics - 3 09-17-17

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Initialization: Be sure the files *NTGStylesheet2.nb* and *NTGUtilityFunctions.m* is are in the same directory as that from which this notebook was loaded. Then execute the cell immediately below by mousing left on the cell bar to the right of that cell and then typing "shift" + "enter". Respond "Yes" in response to the query to evaluate initialization cells.

```
SetDirectory[NotebookDirectory[]];
 (* set directory where source files are located *)
SetOptions[EvaluationNotebook[], (* load the StyleSheet *)
  StyleDefinitions → Get["NTGStylesheet2.nb"]];
Get["NTGUtilityFunctions.m"]; (* Load utilities package *)
```

Purpose

This is the third of four notebooks dealing with electrostatic topics and calculations as treated in Jackson *Classical Electrodynamics.*

This notebook focuses on the expansion of $\frac{1}{|\mathbf{r} - \hat{\mathbf{r}}|}$ in terms of a power series of eigenfunctions, and its application to the specific problem of determining the potential associated with a ring of charge

1 Expansion of $\frac{1}{|\mathbf{r} - \hat{\mathbf{r}}|}$ in terms of Legendre polynomials

I derive the useful expansion

$$\frac{1}{\left|\mathbb{r} - \hat{\mathbb{r}}\right|} = \frac{1}{r_>} \sum_{\ell=0}^{\infty} \left(\frac{r_<}{r_>}\right)^{\ell} \mathbf{LegendreP}[\ell, \ Cos[\zeta]]$$
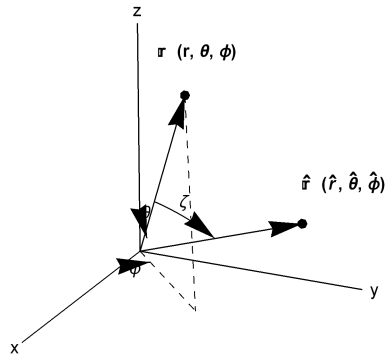
I invoke the Notation package in order to use "physics like" notation.

```
<< Notation`
```

```
Symbolize[ r̂ ];
Symbolize[ r̂ ];
Symbolize[ θ̂ ];
Symbolize[ φ̂ ];
Symbolize[ r> ];
Symbolize[ r< ];
```

Start with

```
w1[1] = ───────1───────
        Norm[ r − r̂ ]
```

$$\frac{1}{\mathbf{Norm}\left[\mathbb{r} - \hat{\mathbb{r}}\right]}$$

Without loss of generality, orient $\mathbb{r}$ is oriented along the z-axis so that ζ becomes the polar angle

To facilitate the valuation of this expression, I introduce a "dot product" operator.

```
Clear[dot]; SetAttributes[dot, Orderless];
dot[a_ + b_, c_ + d_] := dot[a, a] + dot[a, b] + dot[b, a] + dot[b, b];
dot[a_ b_, c_] /; And[FreeQ[a, 𝕝ᵣ], FreeQ[a, 𝕝_z]] := a dot[b, c];
dot[𝕣, 𝕣] := r²; dot[𝕣̂, 𝕣̂] := r̂²;
dot[𝕣, 𝕣̂] := r r̂ Cos[ζ];
```

Then, upon defining Norm in terms of dot

```
w1[2] = w1[1] /. Norm[a_] → √dot[a, a]
```

$$\frac{1}{\sqrt{r^2 + \hat{r}^2 - 2\,r\,\hat{r}\,Cos[\zeta]}}$$

For the case $\hat{r} > r$

```
w1[3] = Simplify[Series[w1[2], {r, 0, 5}], {r > 0, r̂ > r}]
```

$$\frac{1}{\hat{r}} + \frac{Cos[\zeta]\,r}{\hat{r}^2} + \frac{\left(1 + 3\,Cos[2\,\zeta]\right)\,r^2}{4\,\hat{r}^3} + \frac{Cos[\zeta]\,\left(-1 + 5\,Cos[2\,\zeta]\right)\,r^3}{4\,\hat{r}^4} + \frac{1}{64\,\hat{r}^5}$$
$$\left(9 + 20\,Cos[2\,\zeta] + 35\,Cos[4\,\zeta]\right)\,r^4 + \frac{1}{8\,\hat{r}^6}Cos[\zeta]\,\left(15 - 70\,Cos[\zeta]^2 + 63\,Cos[\zeta]^4\right)\,r^5 + O[r]^6$$

Similarly, for $r > \hat{r}$

```
w1[4] = Simplify[Series[w1[2], {r, 0, 5}], {r̂ > 0, r > r̂}]
```

$$\frac{1}{\hat{r}} + \frac{Cos[\zeta]\,r}{\hat{r}^2} + \frac{\left(1 + 3\,Cos[2\,\zeta]\right)\,r^2}{4\,\hat{r}^3} + \frac{Cos[\zeta]\,\left(-1 + 5\,Cos[2\,\zeta]\right)\,r^3}{4\,\hat{r}^4} + \frac{1}{64\,\hat{r}^5}$$
$$\left(9 + 20\,Cos[2\,\zeta] + 35\,Cos[4\,\zeta]\right)\,r^4 + \frac{1}{8\,\hat{r}^6}Cos[\zeta]\,\left(15 - 70\,Cos[\zeta]^2 + 63\,Cos[\zeta]^4\right)\,r^5 + O[r]^6$$

It is traditional to use a short hand rotation

```
w1[5] = w1[3] /. r → r_< /. r̂ → r_>
```

$$\frac{1}{r_>} + \frac{Cos[\zeta]\,r_<}{r_>^2} + \frac{\left(1 + 3\,Cos[2\,\zeta]\right)\,r_<^2}{4\,r_>^3} + \frac{Cos[\zeta]\,\left(-1 + 5\,Cos[2\,\zeta]\right)\,r_<^3}{4\,r_>^4} + \frac{1}{64\,r_>^5}$$
$$\left(9 + 20\,Cos[2\,\zeta] + 35\,Cos[4\,\zeta]\right)\,r_<^4 + \frac{1}{8\,r_>^6}Cos[\zeta]\,\left(15 - 70\,Cos[\zeta]^2 + 63\,Cos[\zeta]^4\right)\,r_<^5 + O[r_<]^6$$

where $r_>$ denotes max[r, $\hat{r}$] and $r_<$ denotes min[r, $\hat{r}$]

Consider the first few Legendre polynomials

```
Table[{Inactive[LegendreP][ℓ, Cos[ξ]], LegendreP[ℓ, Cos[ξ]]}, {ℓ, 0, 5}] //
  ColumnForm
```

{LegendreP[0, Cos[ξ]], 1}
{LegendreP[1, Cos[ξ]], Cos[ξ]}
{LegendreP[2, Cos[ξ]], $\frac{1}{2}$ $\left(-1 + 3\,Cos[\xi]^2\right)$}
{LegendreP[3, Cos[ξ]], $\frac{1}{2}$ $\left(-3\,Cos[\xi] + 5\,Cos[\xi]^3\right)$}
{LegendreP[4, Cos[ξ]], $\frac{1}{8}$ $\left(3 - 30\,Cos[\xi]^2 + 35\,Cos[\xi]^4\right)$}
{LegendreP[5, Cos[ξ]], $\frac{1}{8}$ $\left(15\,Cos[\xi] - 70\,Cos[\xi]^3 + 63\,Cos[\xi]^5\right)$}

Despite the slightly different appearance of the ζ dependence, I suspect that the expansion for general ζ is

$$\frac{1}{|\mathbf{r} - \hat{\mathbf{r}}|} = \frac{1}{r_>}\sum_{\ell=0}^{\infty}\left(\frac{r_<}{r_>}\right)^{\ell}\text{LegendreP}[\ell,\ Cos[\zeta]]$$

I check this hypothesis against the result derived above.

```
w1[6] = 1/r_> Sum[(r_</r_>)^ℓ LegendreP[ℓ, Cos[ξ]], {ℓ, 0, 5}] // Expand;

w1[6] = Collect[w1[6], {r_<, r_>}]
```

$\frac{1}{r_>} + \frac{r_<\,Cos[\xi]}{r_>^2} + \frac{r_<^2\left(-\frac{1}{2} + \frac{3\,Cos[\xi]^2}{2}\right)}{r_>^3} + \frac{r_<^3\left(-\frac{3\,Cos[\xi]}{2} + \frac{5\,Cos[\xi]^3}{2}\right)}{r_>^4} +$

$\frac{r_<^4\left(\frac{3}{8} - \frac{15\,Cos[\xi]^2}{4} + \frac{35\,Cos[\xi]^4}{8}\right)}{r_>^5} + \frac{1}{r_>^6}r_<^5\left(\frac{15\,Cos[\xi]}{8} - \frac{35\,Cos[\xi]^3}{4} + \frac{63\,Cos[\xi]^5}{8}\right)$

Compare the two expressions

```
w1[7] = Collect[Expand[Normal@w1[5] - w1[6]], {r_<, r_>}]
```

$\frac{r_<^2\left(\frac{3}{4} - \frac{3\,Cos[\xi]^2}{2} + \frac{3}{4}\,Cos[2\,\xi]\right)}{r_>^3} + \frac{1}{r_>^4}r_<^3\left(\frac{5\,Cos[\xi]}{4} - \frac{5\,Cos[\xi]^3}{2} + \frac{5}{4}\,Cos[\xi]\,Cos[2\,\xi]\right) +$

$\frac{1}{r_>^5}r_<^4\left(-\frac{15}{64} + \frac{15\,Cos[\xi]^2}{4} - \frac{35\,Cos[\xi]^4}{8} + \frac{5}{16}\,Cos[2\,\xi] + \frac{35}{64}\,Cos[4\,\xi]\right)$
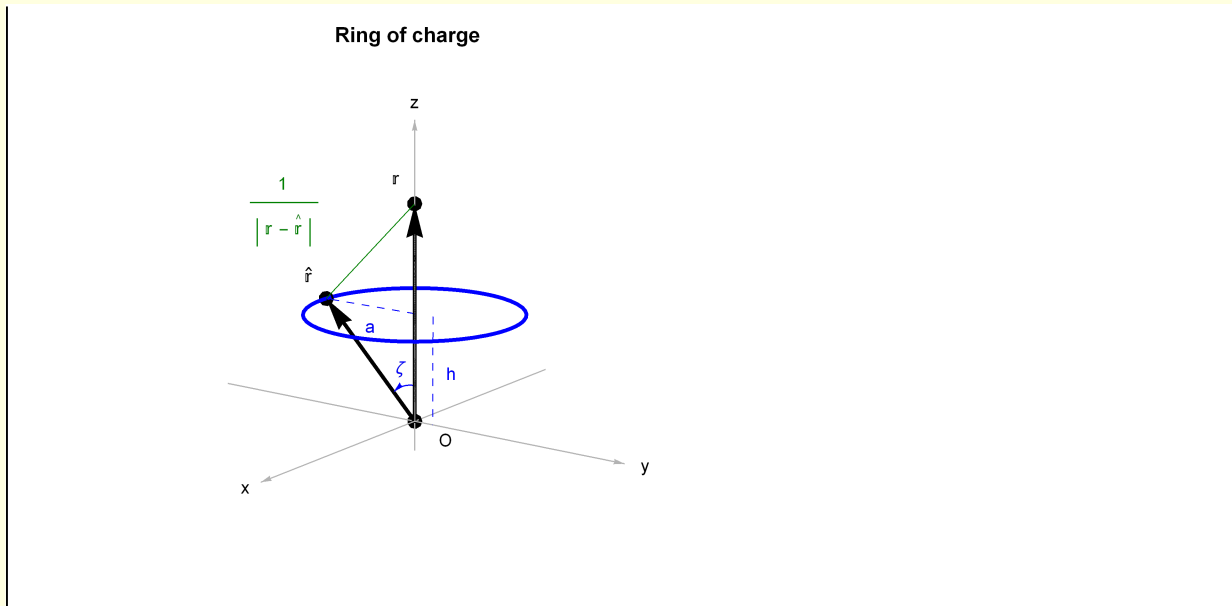
Rather than trying to be clever with trig identities, I let Mathematica do the work

```
w1[8] = Simplify[#, TrigReduce] & /@ w1[7]
```

0

and my supposition is confirmed.

2 Application of expansion to ring of charge

Ring of charge

The potential on the z-axis associated with a differential element of charge on the ring is

$$d\Phi(r = z) = \frac{dq}{\sqrt{r^2 + d^2 - 2\, r\, d\, \cos[\zeta]}}$$

$$d = \sqrt{a^2 + h^2}, \quad \tan[\zeta] = a/h$$

For uniform azimuthal charge density, σ = q/(2 π), integration over the azimuthal angle ϕ results in

$$\Phi(r = z) = \frac{q}{\sqrt{r^2 + d^2 - 2\, r\, d\, \cos[\zeta]}}$$

which has the expansion

$$\Phi(r = z) = \frac{1}{\max(r,\ d)} \sum_{\ell=0}^{\infty} \left(\frac{\min(r,\ d)}{\max(r,\ d)} \right)^{\ell} \text{LegendreP}[\ell,\ \cos[\zeta]]$$

Also, for problems with azimuthal symmetry, we have in general that

$$\Phi(r,\ \theta) = \sum_{\ell=0}^{\infty} \left(A_\ell\, r_<^\ell + B_\ell\, r_>^{-(\ell+1)} \right) P_\ell(\cos(\theta)) = \sum_{\ell=0}^{\infty} \Phi(r = z)\, P_\ell(\cos(\theta))$$

I define the following function

```
Clear[ΦExpansion];
ΦExpansion[r_, θ_, q_, a_, h_, ℓMax_] :=
 Module[{rhat = √(a² + h²) , ξ = ArcTan[a/h], rMin, rMax},
  rMin = Min[r, rhat]; rMax = Max[r, rhat];
   ℓMax
   ∑   rMinℓ
  ℓ=0 ───────── LegendreP[ℓ, Cos[ξ]] LegendreP[ℓ, Cos[θ]]]
       rMaxℓ⁺¹
```

Clearly, the accuracy of this expansion will depend on how many terms are included in the sum. I compare the exact on-axis result with expansion results for ℓMax = 10, 20, 30. Accuracy is hard to achieve near z = $\hat{r}$.

```
Module[{q = 1, a = 1, h = 1, θ = 0, r̂, θ̂, lab, G},
 r̂ = √(a² + h²) ; θ̂ = ArcTan[a/h];

 lab = Stl["Comparing expansion vs exact on-axis potential"];
                           q
 G[1] = Plot[Callout[ ─────────────────────── , ""], {r, 0, 3 r̂}, Epilog →
                     √(r² + r̂² - 2 r r̂ Cos[θ̂])
    {Directive[Black, Dashed], Line[{{r̂, 0}, {r̂, 1000}}], Text["z = r̂", {r̂, 0.4}]},
    AxesLabel → {Stl["z"], Stl["Φ"]}, PlotLabel → lab];
 G[2] = Plot[Evaluate@Table[ΦExpansion[r, θ, q, a, h, ℓMax], {ℓMax, 10, 30, 10}],
    {r, 0, 3 r̂}, PlotStyle → {Blue, Green, Red}, Epilog →
     {Directive[Black, Dashed], Line[{{r̂, 0}, {r̂, 1000}}], Text["z = r̂", {r̂, 0.4}]},
    AxesLabel → {Stl["z"], Stl["Φ"]}, PlotLabel → lab,
    PlotLegends → {"ℓMax = 10", "ℓMax = 20", "ℓMax = 30"}];
 Show[G[2], G[1]]]
```
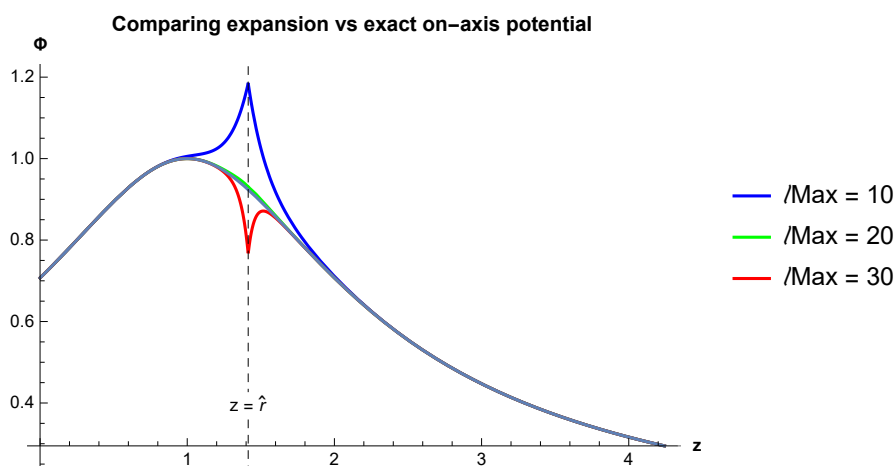

More detail about the discrepancy between the series expansion and the exact on-axis result can be elucidated by considering

$$\text{discrepancy} \equiv \left| \sum_{\ell=0}^{\ell \text{Max}} \Phi[r] \, P_\ell[\cos[0]] - \frac{q}{\sqrt{r^2 + \hat{r}^2 - 2 \, r \, \hat{r} \, \cos[\hat{\theta}]}} \right|$$

```
Module[
 {q = 1, a = 1, h = 1, θ = 0, ℓMax = 100, r̂, θ̂, r, results, lab, Discrepancy, G},
 r̂ = √(a² + h²) ; θ̂ = ArcTan[a / h];
 Discrepancy[r_, ℓMax_] :=
  N@Log@Abs[ΦExpansion[r, θ, q, a, h, ℓMax] - q / (√(r² + r̂² - 2 r r̂ Cos[θ̂]))];
 lab = Stl["log(discrepancy) for r = r̂ (Red) and r = 0.95 r̂ (Blue)"];
 r = r̂;
 results[1] = Table[{ℓMax, Discrepancy[r, ℓMax]}, {ℓMax, 5, ℓMax, 1}];
 r = 0.95 r̂;
 results[2] = Table[{ℓMax, Discrepancy[r, ℓMax]}, {ℓMax, 5, ℓMax, 1}];

 ListPlot[{results[1], results[2]},
  PlotRange → All, PlotLabel → lab, PlotStyle → {Red, Blue},
  AxesLabel → {Stl["ℓMax"], Stl["log(discrepancy)"]}, Joined → True]]
```
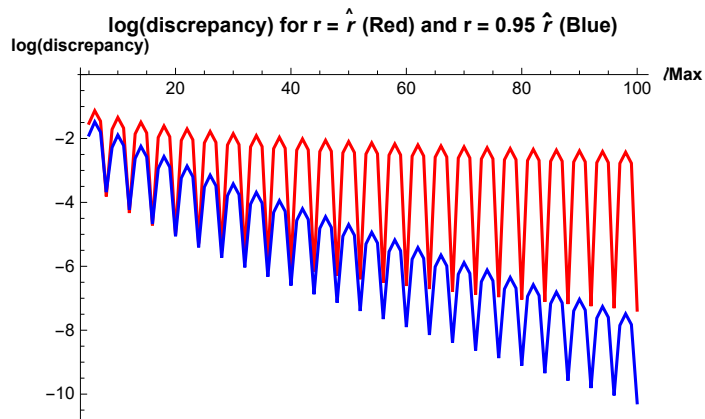

To test off-axis cases, I develop a numerical solution.

Define a function that generates the potential associated with a differential element of the ring.

```
Clear[dΦ];
(* Note the use of : required because r̂ etc have been symbolized *)
dΦ[{r_, θ_, φ_}, {r̂ : _, θ̂ : _, φ̂ : _}, q_] :=
 Module[{𝕣, 𝕣̂, dΦ, σ = q / (2 π)},
  𝕣 = r {Sin[θ] Cos[φ], Sin[θ] Sin[φ], Cos[θ]};
  𝕣̂ = r̂ {Sin[θ̂] Cos[φ̂], Sin[θ̂] Sin[φ̂], Cos[θ̂]};
  dΦ = σ / Norm[𝕣 - 𝕣̂]]
```

Then, the potential is obtain by performing a quadrature over φ̂

```
Clear[ΦQuadrature];
ΦQuadrature[{r_, θ_, φ_}, {r̂ : _, θ̂ : _}, q_] :=
 NIntegrate[dΦ[{r, θ, φ}, {r̂, θ̂, φ̂}, q], {φ̂, 0, 2 π}]
```

I generate a contour plot of the potential in one quadrant of the x-z plane. The calculation takes a while,

and there are complaints when the contour plotting package asks for some evaluations.

```
Timing@Module[{ε = 0.01, q = 1, a = 1, h = 1, ϕ = 0, r̂, θ̂, lab},
  r̂ = √(a² + h²) ; θ̂ = ArcTan[a / h];
  lab = Stl["Φ by numerical quadrature"];
  ContourPlot[ΦQuadrature[{√(x² + z²) , ArcTan[x / z], ϕ}, {r̂, θ̂}, q], {x, 0 + ε, 2 + ε},
   {z, 0 + ε, 2 + ε}, FrameLabel → {{Rotate[Stl["z"], π / 2], ""}, {Stl["x"], lab}},
   ImageSize → 400, ColorFunction → "DeepSeaColors"]]
```
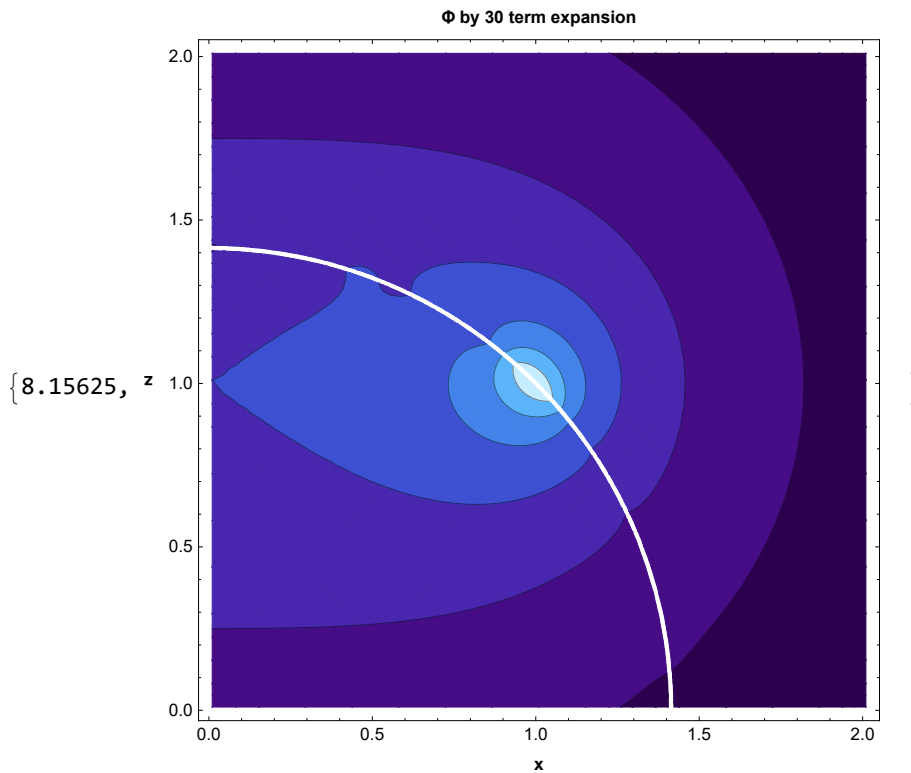
{29.5,

Φ by numerical quadrature

}
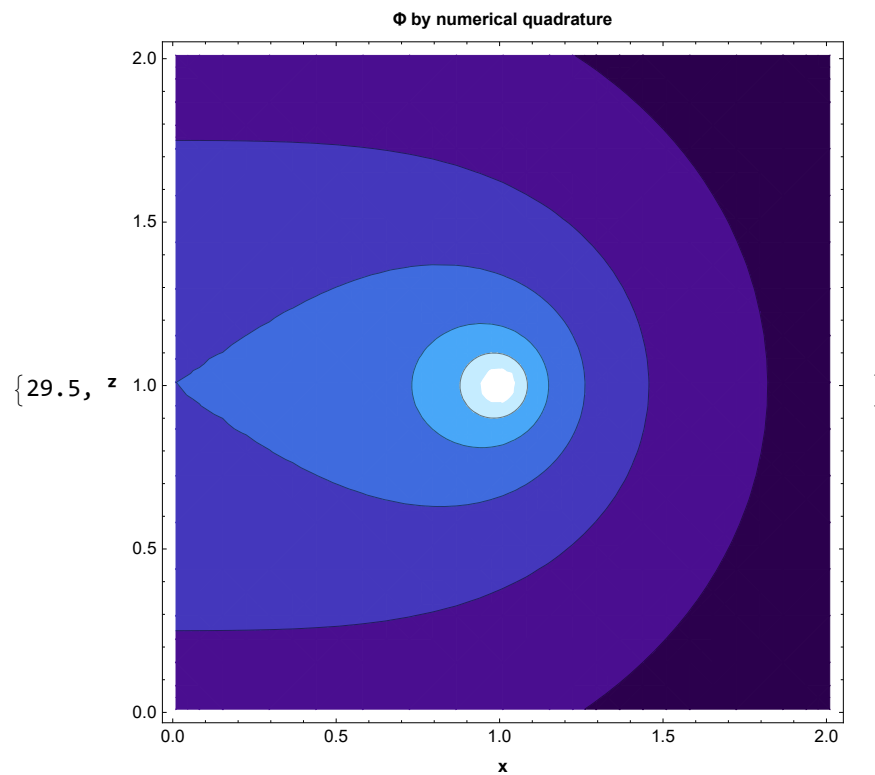
I repeat the calculation using the approximate expansion with /Max = 30.

```
Timing@Module[{ε = 0.01, q = 1, a = 1, h = 1, φ = 0, ℓMax = 30, r̂, θ̂, lab},

  r̂ = √(a² + h²) ; θ̂ = ArcTan[a / h];
  lab = Stl["Φ by 30 term expansion"];
  ContourPlot[ΦExpansion[√(x² + z²) , ArcTan[x / z], q, a, h, ℓMax], {x, 0 + ε, 2 + ε},
   {z, 0 + ε, 2 + ε}, FrameLabel → {{Rotate[Stl["z"], π / 2], ""}, {Stl["x"], lab}},
   ImageSize → 400, ColorFunction → "DeepSeaColors"]]
```

$\{8.15625,$ $\}$

As expected, the expansion breaks down near r = $\hat{r}$, which is $\sqrt{2}$ for the chosen parameters.

Visualizations

```
Clear[ShowCoordinateSystemForExpansion];
ShowCoordinateSystemForExpansion[imageSize_] :=
  Module[{δ = 0.05, xMax = 1.5, O = {0, 0, 0}, axes, θarc, ϕarc, ξarc, r, rh,
    rprojection, OP, liner, linerh, lineBase, range3D , G, StoC, PointOnArc},
   StoC[r_, θ_, ϕ_] := {r Sin[θ] Cos[ϕ], r Sin[θ] Sin[ϕ], r Cos[θ] };
   (* point on the arc connecting points P1 and P2*)
   PointOnArc[α_(*0<α<1*), P1_, P2_, rMag_(*distance of arc from O*)] :=
    Module[{Pα = P1 + α (P2 - P1)}, rMag Pα / Norm[Pα]];

   range3D = {{-1.1 xMax, xMax}, {-1.1 xMax, xMax}, {-1.1 xMax, xMax}};


   r = StoC[2, π/4, π/5];
   rprojection = StoC[1, π/2, π/4];
   rh = StoC[1.5, 3 π/8, 3 π/8] ;


   axes = {{Line[{{0, 0, 0}, {1.5, 0, 0}}], Text["x", 1.05 {1.5, 0, 0} ]},
     {Line[{{0, 0, 0}, {0, 1.5, 0}}], Text["y", 1.05 {0, 1.5, 0} ]},
     {Line[{{0, 0, 0}, {0, 0, 1.5}}], Text["z", 1.05 {0, 0, 1.5} ]}};

   liner = {Black, Arrowheads[Small], Arrow[{O, r}],
     PointSize[0.015], Point[r], Text[Stl["r   (r, θ, ϕ)"], 1.2 r]};
   linerh = {Black, Arrowheads[Small], Arrow[{O, rh}], PointSize[0.015],
     Point[rh], Text[Stl["r̂   (r̂, θ̂, ϕ̂)"], 1.2 rh + {0, 0, 0.2}]};
   lineBase = {Dashed, Line[{r, rprojection}], Line[{O, rprojection}]};
   θarc = With[{θVals = StoC[0.2, #, π / 4] & /@ Range[0, π / 4, π / 64]},
     {Black, Arrowheads[Small], Arrow[θVals], Text["θ", 1.6 θVals〚8〛]}];
   ϕarc = With[{ϕVals = StoC[0.2, π/2, #] & /@ Range[0, π / 4, π / 64]},
     {Black, Arrowheads[Small], Arrow[ϕVals], Text["ϕ", 1.6 ϕVals〚8〛]}];
   ξarc = With[{ξvals = PointOnArc[#, r, rh, 0.75] & /@ Range[0, 1, 0.1]},
     {Black, Arrowheads[Small], Arrow[ξvals], Text["ξ", 1.2 ξvals〚5〛]}];

   G[1] = Graphics3D[{axes, {θarc, ϕarc, ξarc}, { liner, lineBase, linerh}},
     Boxed → False, PlotRange → range3D, ViewPoint → {2.2, 1.1, 1.0},
     ImagePadding → 1, PlotRangePadding → None]];
With[{imageSize = 300},
  [imageSize]]
```
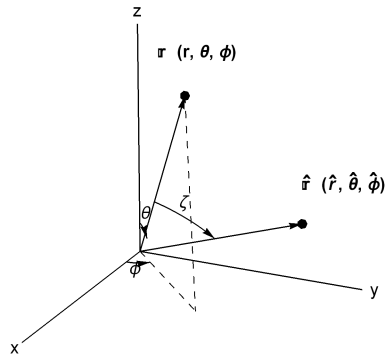
z

r (r, θ, ϕ)

$\hat{\mathbf{r}}$ ($\hat{r}$, $\hat{\theta}$, $\hat{\phi}$)

θ

ζ

ϕ

y

x

```
Module[{δ = 0.2, vp = {3.5, 2.5, 1}, xMax = 2, a = 1, h = 1, axes, O,
   X, Y, Z, r, r̂, pO, pr, prh, ar, arh, lrhr, la, lh, ζarc, lab, StoC, G},
  StoC[r_, θ_, ϕ_] := {r Sin[θ] Cos[ϕ], r Sin[θ] Sin[ϕ], r Cos[θ] };
  {O, X, Y, Z} = {{0, 0, 0}, {2, 0, 0}, {0, 2, 0}, {0, 0, 2.5}};
  axes = {GrayLevel[0.7], Arrowheads[Small],
    {Arrow[1.1 {-X, X}], Arrow[1.1 {-Y, Y}], Arrow[1.1 {-0.1 Z, Z}]},
    {Black, {Text["x", X + {2 δ, 0, 0}, {0, 0}], Text["y", Y + {0, 2 δ, 0}, {0, 0}],
      Text["z", (Z + {0, 0, 2 δ}), {0, 0}]}}};

  (* Observation point z = 2 above origin *)
  r = {0, 0, 2};
  (* dϕ element location *)
  r̂ = {0, -a, h};

  pO = {PointSize[0.02], Point[O], Text["O", O + {-δ, δ, -δ}]};
  pr = {PointSize[0.02], Point[r], Text["r", r + {0, -δ, δ}]};
  prh = {PointSize[0.02], Point[r̂], Text["r̂", r̂ + {0, -δ, δ}]};
  ar = {Directive[Black, Thickness[0.005]], Arrow[{O, r}]};
  arh = {Directive[Black, Thickness[0.005]], Arrow[{O, r̂}]};
  la = {Directive[Blue, Dashed],
    Line[{{0, 0, a}, {0, -a, a}}], Text["a", {0, -a/2, h - δ}] };
  lh = {Directive[Blue, Dashed], Line[{{0, δ, 1}, O + {0, δ, 0}}],
    Text["h", {0, 2 δ, h/2}] };
  lrhr = {Darker[Green, 0.5], Line[{r̂, r}], Text["    1    ", {0, -1.5, 1.75}] };
                                             | r - r̂ |
  ζarc = {Blue, Arrowheads[Small], Arrow@Table[StoC[a/3, θ, π/2], {θ, 0, -π/4, -π/64}],
    Text["ζ", StoC[a/1.75, π/8, 0]]};

  lab = Stl["Ring of charge"];
  G[1] = Graphics3D[{la, lh, pO, pr, prh, ar, arh, lrhr, ζarc, axes},
    Boxed → False, ImageSize → 400, ViewPoint → vp, PlotLabel → lab];

  (* the ring *)
  G[2] = ParametricPlot3D[{{a Cos[ϕ], a Sin[ϕ], h }},
    {ϕ, 0, 2 π}, Axes → None, Boxed → False, ViewPoint → vp, PlotRange →
     {{-xMax, xMax}, {-xMax, xMax}, {-xMax, xMax}}, PlotStyle → {Thick, Blue}];
  Show[G[1], G[2]]]
```

Ring of charge