TB 13-17 Entropy 03-24-18

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Initialization: Be sure the file *NTGUtilityFunctions.m* is in the same directory as that from which this notebook was loaded. Then execute the cell immediately below by mousing left on the cell bar to the right of that cell and then typing "shift" + "enter". Respond "Yes" in response to the query to evaluate initialization cells.

```
SetDirectory[NotebookDirectory[]];
(* set directory where source files are located *)
Get["NTGUtilityFunctions.m"]; (* Load utilities package *)
```

Purpose

This is the 10th in a series of notebooks in which I work through material and exercises in the magisterial new book *Modern Classical Physics* by Kip S. Thorne and Roger D. Blandford. If you are a physicist of any ilk, BUY THIS BOOK. You will learn from a close reading and from solving the exercises.

Exercise 13.17 *Derivation: Entropy Increase*

(a) Derive the Lagrangian equation (13.74) for the rate of increase of entropy in a dissipative fluid by the steps in the sentence preceeding that equation. [Hints: If you have already done the analogous problem, Ex. 13.13, for an ideal fluid, then you need only compute the new terms that arise from the dissipative momentum flux $\mathbf{T}_{\text{vis}} = -\zeta\theta\mathbf{g} - 2\eta\boldsymbol{\sigma}$ and dissipative energy fluxes $\mathbf{F}_{\text{vis}} = \mathbf{T}_{\text{vis}} \cdot \mathbf{v}$ and $\mathbf{F}_{\text{cond}} = -\kappa\boldsymbol{\nabla}T$. The sum of these new contributions, when you subtract $\mathbf{v} \cdot$ (momentum conservation) from energy conservation, is $\boldsymbol{\nabla} \cdot \mathbf{F}_{\text{cond}} + \boldsymbol{\nabla} \cdot (\mathbf{T}_{\text{vis}} \cdot \mathbf{v}) - \mathbf{v} \cdot (\boldsymbol{\nabla} \cdot \mathbf{T}_{\text{vis}})$; and this must be added to the left side of the result $\rho T ds/dt = 0$, Eq. (13.58), for an ideal fluid. In doing the algebra, it may be useful to decompose the gradient of the velocity into its irreducible tensorial parts, Eq. (13.66).]

(b) From the Lagrangian equation of entropy increase (13.74) derive the corresponding Eulerian equation (13.75).

The solution of this problem requires quite elaborate calculations. I had to review several topics to perform the required steps. I followed calculations in Landau and Lifshitz *Fluid Mechanics.*

I will approach this calculation in stages.

Base calculation with no dissipative effects

The energy/unit volume of a fluid is

$$\rho\left(\frac{v^2}{2} + e\right) \tag{1}$$

where the first term is the kinetic energy associated with the fluid motion and the second term is the internal energy of the fluid, e.g., the energy associated with the thermal motion of fluid particles.

I calculate an expression for the conservation of energy

$$
\begin{aligned}
\frac{\partial \rho\left(\frac{v^2}{2} + e\right)}{\partial t} &= \frac{1}{2}\frac{\partial \rho v^2}{\partial t} + \frac{\partial \rho e}{\partial t} \\
&= \frac{\rho}{2}\frac{\partial v^2}{\partial t} + \frac{v^2}{2}\frac{\partial \rho}{\partial t} + \frac{\partial \rho e}{\partial t} \\
&= \rho\, \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + \frac{v^2}{2}\frac{\partial \rho}{\partial t} + \frac{\partial \rho e}{\partial t}
\end{aligned}
\tag{2}
$$

Make use of the conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \mathbf{v}) = 0 \tag{3}$$

and Euler's equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \cdot \mathbf{v} = -\frac{\nabla P}{\rho} \tag{4}$$

to rewrite (2) as

$$\frac{\partial \rho\left(\frac{v^2}{2} + e\right)}{\partial t} = \rho\, \mathbf{v} \cdot \left(-(\mathbf{v} \cdot \nabla) \cdot \mathbf{v} - \frac{\nabla P}{\rho}\right) + \frac{v^2}{2}(-\nabla \cdot (\rho\, \mathbf{v})) + \frac{\partial \rho e}{\partial t} \tag{5}$$

I first calculate the terms independent of the internal energy e.
Note

$$-(\mathbf{v} \cdot \nabla) \cdot \mathbf{v} = -\frac{1}{2}\nabla(\mathbf{v} \cdot \mathbf{v}) = -\nabla\frac{v^2}{2} \tag{6}$$

$$
\begin{aligned}
\rho\, \mathbf{v} \cdot \left(-(\mathbf{v} \cdot \nabla) \cdot \mathbf{v} - \frac{\nabla P}{\rho}\right) + \frac{v^2}{2}(-\nabla \cdot (\rho\, \mathbf{v})) &= \rho\, \mathbf{v} \cdot \left(-\nabla\frac{v^2}{2} - \frac{\nabla P}{\rho}\right) + \frac{v^2}{2}(-\nabla \cdot (\rho\, \mathbf{v})) \\
&= -\rho\, \mathbf{v} \cdot \left(\nabla\frac{v^2}{2}\right) - \frac{v^2}{2}(\nabla \cdot (\rho\, \mathbf{v})) - \rho\, \mathbf{v} \cdot \left(\frac{\nabla P}{\rho}\right)
\end{aligned}
\tag{7}
$$

$$= -\nabla \cdot \left(\rho \, \boldsymbol{v} \, \frac{v^2}{2} \right) - \rho \, \boldsymbol{v} \cdot \left(\frac{\nabla P}{\rho} \right)$$

From thermodynamics we have for the enthalpy

$$h = e + \frac{P}{\rho}$$

$$dh = T \, ds + \frac{dP}{\rho} \tag{8}$$

$$\nabla h = T \, \nabla s + \frac{\nabla P}{\rho}$$

Using (6) - (8)

$$\rho \, \boldsymbol{v} \cdot \left(-(\boldsymbol{v} \cdot \nabla) \cdot \boldsymbol{v} - \frac{\nabla P}{\rho} \right) + \frac{v^2}{2} \left(-\nabla \cdot (\rho \, \boldsymbol{v}) \right) = -\nabla \cdot \left(\rho \, \boldsymbol{v} \, \frac{v^2}{2} \right) - \rho \, \boldsymbol{v} \cdot \left(\frac{\nabla P}{\rho} \right)$$

$$= -\nabla \cdot \left(\rho \, \boldsymbol{v} \, \frac{v^2}{2} \right) - \rho \, \boldsymbol{v} \cdot (\nabla h - T \, \nabla s) \tag{9}$$

Now consider

$$\frac{\partial \rho \, e}{\partial t} = e \, \frac{\partial \rho}{\partial t} + \rho \, \frac{\partial e}{\partial t} \tag{10}$$

Again from thermodynamics

$$de = T \, ds - P \, dV = T ds + \frac{P}{\rho^2} \, d\rho$$

$$\frac{\partial e}{\partial t} = T \, \frac{\partial s}{\partial t} + \frac{P}{\rho^2} \, \frac{\partial \rho}{\partial t} \tag{11}$$

So

$$\frac{\partial \rho \, e}{\partial t} = e \, \frac{\partial \rho}{\partial t} + \rho \left(T \, \frac{\partial s}{\partial t} + \frac{P}{\rho^2} \, \frac{\partial \rho}{\partial t} \right)$$

$$= \left(e + \frac{P}{\rho} \right) \frac{\partial \rho}{\partial t} + \rho \, T \, \frac{\partial s}{\partial t} \tag{12}$$

$$= h \, \frac{\partial \rho}{\partial t} + \rho \, T \, \frac{\partial s}{\partial t}$$

There are no dissipative forces so

$$\frac{ds}{dt} = 0 = \frac{\partial s}{\partial t} + (\boldsymbol{v} \cdot \nabla) s \tag{13}$$

Using (13) and the continuity of mass

$$
\begin{aligned}
\frac{\partial \rho e}{\partial t} &= h \frac{\partial \rho}{\partial t} + \rho T \frac{\partial s}{\partial t} \\[2mm]
&= h(-\nabla \cdot (\rho\, \boldsymbol{v})) + \rho T (-(\boldsymbol{v} \cdot \nabla)s) \\[2mm]
&= -h \nabla \cdot (\rho\, \boldsymbol{v}) - \rho T (\boldsymbol{v} \cdot \nabla)s
\end{aligned}
\tag{14}
$$

Combining the two sub-calculations (9) and (14)

$$
\begin{aligned}
\frac{\partial \rho\left(\frac{v^2}{2} + e\right)}{\partial t} &= \frac{1}{2} \frac{\partial \rho v^2}{\partial t} + \frac{\partial \rho e}{\partial t} \\[2mm]
&= -\nabla \cdot \left(\rho\, \boldsymbol{v}\, \frac{v^2}{2}\right) - \rho\, \boldsymbol{v} \cdot (\nabla h - T \nabla s) \\[2mm]
&\quad - h \nabla \cdot (\rho\, \boldsymbol{v}) - \rho T (\boldsymbol{v} \cdot \nabla)s \\[2mm]
&= -\nabla \cdot \left(\rho\, \boldsymbol{v}\, \frac{v^2}{2}\right) - \rho\, \boldsymbol{v} \cdot \nabla h \; - h \nabla \cdot (\rho\, \boldsymbol{v}) \\[4mm]
&= -\nabla \cdot \left(\rho\, \boldsymbol{v}\, \frac{v^2}{2}\right) - \nabla \cdot \rho\, \boldsymbol{v}\, h \\[2mm]
&= -\nabla \cdot \left(\rho\, \boldsymbol{v}\left(\frac{v^2}{2} + h\right)\right)
\end{aligned}
\tag{15}
$$

Finally, the conservation of energy is

$$
\frac{\partial \rho\left(\frac{v^2}{2} + e\right)}{\partial t} = -\nabla \cdot \left(\rho\, \boldsymbol{v}\left(\frac{v^2}{2} + h\right)\right)
\tag{16}
$$

The conservation of energy is

$$
\frac{\partial \rho\left(\frac{v^2}{2} + e\right)}{\partial t} = -\nabla \cdot \left(\rho\, \boldsymbol{v}\left(\frac{v^2}{2} + h\right)\right)
$$

Proceed further to understand the role of the different terms. If the enthalpy is expanded

$$
\frac{\partial \rho\left(\frac{v^2}{2} + e\right)}{\partial t} = -\nabla \cdot \left(\rho\, \boldsymbol{v}\left(\frac{v^2}{2} + e + \frac{P}{\rho}\right)\right)
\tag{17}
$$

$$-\nabla \cdot \left(\rho\, \boldsymbol{v} \left(\frac{v^2}{2} + e \right) + \boldsymbol{v}\,P \right)$$

Consider a "control volume $\mathcal{V}$ with surface area $\mathcal{A}$

$$\int_{\mathcal{V}} \mathrm{d}\mathcal{V}\, \frac{\partial \rho \left(\frac{v^2}{2} + e \right)}{\partial t} = - \int_{\mathcal{V}} \mathrm{d}\mathcal{V} \left(\nabla \cdot \left(\rho\, \boldsymbol{v} \left(\frac{v^2}{2} + e \right) + \boldsymbol{v}\,P \right) \right) \tag{18}$$

or

$$\frac{\partial}{\partial t} \int_{\mathcal{V}} \mathrm{d}\mathcal{V}\, \rho \left(\frac{v^2}{2} + e \right) = - \int_{\mathcal{A}} \mathrm{d}\mathcal{A}\, \mathbb{1}_n \cdot \left(\rho\, \boldsymbol{v} \left(\frac{v^2}{2} + e \right) + \boldsymbol{v}\,P \right)$$

$$= - \int_{\mathcal{A}} \mathrm{d}\mathcal{A}\, \mathbb{1}_n \cdot \left(\rho\, \boldsymbol{v} \left(\frac{v^2}{2} + e \right) \right) - \int_{\mathcal{A}} \mathrm{d}\mathcal{A}\, \mathbb{1}_n \cdot (\boldsymbol{v}\,P) \tag{19}$$

The first term on the right hand side is the flux of internal and kinetic energy through the surface of $\mathcal{V}$. The second term is the work done on the surface by the fluid pressure.

Inclusion of viscosity and thermal conductivity

The conservation of energy derived above (16) was

$$\frac{\partial \rho \left(\frac{v^2}{2} + e \right)}{\partial t} = - \nabla \cdot \left(\rho\, \boldsymbol{v} \left(\frac{v^2}{2} + h \right) \right) \tag{20}$$

where the term on the rhs represents the energy flux. If we add the additional effects of viscosity and thermal conductivity

$$\frac{\partial \rho \left(\frac{v^2}{2} + e \right)}{\partial t} = - \nabla \cdot \left(\rho\, \boldsymbol{v} \left(\frac{v^2}{2} + h + \boldsymbol{v} \cdot \overset{\leftrightarrow}{\sigma} - \kappa \nabla T \right) \right) \tag{21}$$

The conservation of energy is conservative form is

$$\frac{\partial \rho \left(\frac{v^2}{2} + e \right)}{\partial t} = - \nabla \cdot \left(\rho\, \boldsymbol{v} \left(\frac{v^2}{2} + h + \boldsymbol{v} \cdot \overset{\leftrightarrow}{\sigma} - \kappa \nabla T \right) \right)$$

It is traditional to represent this equation in a different form.

Reconsider the expansion of the lhs but, instead of the Euler equation, use the Navier Stokes equation.

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla) \cdot \boldsymbol{v} = - \frac{\nabla P}{\rho} + \frac{1}{\rho} \frac{\partial \sigma_{ik}}{\partial x_k} \tag{22}$$

In the following I will follow Landau and Lifshitz and use tensor index notation when it makes the details more clear.

Proceeding as above

$$
\frac{\partial \rho \left(\frac{v^2}{2} + e \right)}{\partial t} = \rho \, \boldsymbol{v} \cdot \frac{\partial \, \boldsymbol{v}}{\partial t} + \frac{v^2}{2} \frac{\partial \rho}{\partial t} + \frac{\partial \rho \, e}{\partial t}
$$

$$
= \rho \boldsymbol{v} \cdot \left(-(\boldsymbol{v} \cdot \nabla) \cdot \boldsymbol{v} - \frac{\nabla P}{\rho} + \frac{1}{\rho} \frac{\partial \, \sigma_{ik}}{\partial x_k} \right) + \frac{v^2}{2} \left(-\nabla \cdot (\rho \, \boldsymbol{v}) \right) + \frac{\partial \rho \, e}{\partial t}
$$

$$
= \rho \, \boldsymbol{v} \cdot \left(-(\boldsymbol{v} \cdot \nabla) \cdot \boldsymbol{v} - \frac{\nabla P}{\rho} \right) + v_i \frac{\partial \, \sigma_{ik}}{\partial x_k} - \frac{v^2}{2} \nabla \cdot (\rho \, \boldsymbol{v}) + e \frac{\partial \rho}{\partial t} + \rho \frac{\partial e}{\partial t}
$$

(23)

Note

$$
e \frac{\partial \rho}{\partial t} = -e \, \nabla \cdot (\rho \, \boldsymbol{v})
$$

(24)

$$
\rho \frac{\partial e}{\partial t} = \rho \left(T \frac{\partial s}{\partial t} + \frac{P}{\rho^2} \frac{\partial \rho}{\partial t} \right) = \rho T \frac{ds}{dt} - \frac{P}{\rho} \nabla \cdot (\rho \, \boldsymbol{v}) \Big)
$$

$$
= \rho T \left(\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \nabla s \right) - \frac{P}{\rho} \nabla \cdot (\rho \boldsymbol{v}) \Big)
$$

(25)

$$
v_i \frac{\partial \, \sigma_{ik}}{\partial x_k} = \frac{\partial}{\partial x_k} (v_i \, \sigma_{ik}) - \sigma_{ik} \frac{\partial v_i}{\partial x_k} = \nabla \cdot \left(\boldsymbol{v} \cdot \overset{\leftrightarrow}{\sigma} \right) - \sigma_{ik} \frac{\partial v_i}{\partial x_k}
$$

(26)

With these expressions, it follows that

$$
\frac{\partial \rho \left(\frac{v^2}{2} + e \right)}{\partial t} = -\nabla \cdot \left(\rho \, \boldsymbol{v} \left(\frac{v^2}{2} + h + \boldsymbol{v} \cdot \overset{\leftrightarrow}{\sigma} \right) \right) - \sigma_{ik} \frac{\partial v_i}{\partial x_k} + \rho T \left(\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \nabla s \right)
$$

(27)

Add and subtracting $\nabla \cdot (\kappa \, \nabla T)$ to the right hand side

$$
\frac{\partial \rho \left(\frac{v^2}{2} + e \right)}{\partial t} = -\nabla \cdot \left(\rho \, \boldsymbol{v} \left(\frac{v^2}{2} + h + \boldsymbol{v} \cdot \overset{\leftrightarrow}{\sigma} - \kappa \nabla T \right) \right) - \sigma_{ik} \frac{\partial v_i}{\partial x_k} + \rho T \left(\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \nabla s \right) - \nabla \cdot (\kappa \nabla T)
$$

(28)

Compare this with the starting form for the conservation of energy including viscosity and thermal conductivity and note that the lhs cancels the first term on the rhs. The result is

$$
\rho T \left(\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \nabla s \right) = \sigma_{ik} \frac{\partial v_i}{\partial x_k} + \nabla \cdot (\kappa \nabla T)
$$

(29)

In Landau and Lifshitz §49, this is called the ***general equation of heat transfe***r.

$$\rho T \left(\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \nabla s \right) = \sigma_{ik} \frac{\partial v_i}{\partial x_k} + \nabla \cdot (\kappa \nabla T)$$

This expression can be still further developed.

Use a traditional representation of viscosity to write

$$\sigma_{ik} \frac{\partial v_i}{\partial x_k} = \eta \frac{\partial v_i}{\partial x_k} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_i}{\partial x_i} \right) + \zeta \frac{\partial v_i}{\partial x_k} \delta_{ik} \frac{\partial v_i}{\partial x_i} \tag{30}$$

where η and ζ are defined to be positive.
Then

$$\zeta \frac{\partial v_i}{\partial x_k} \delta_{ik} \frac{\partial v_i}{\partial x_i} = \zeta \frac{\partial v_i}{\partial x_i} \frac{\partial v_i}{\partial x_i} = \zeta (\nabla \cdot \boldsymbol{v})^2 \tag{31}$$

Also, although it is not immediately obvious,

$$\eta \frac{\partial v_i}{\partial x_k} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_i}{\partial x_i} \right) = \frac{\eta}{2} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_i}{\partial x_i} \right)^2 \tag{32}$$

Then

$$\rho T \left(\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \nabla s \right) = \nabla \cdot (\kappa \nabla T) + \frac{\eta}{2} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_i}{\partial x_i} \right)^2 + \zeta \left(\nabla \cdot \vec{v} \right)^2 \tag{33}$$

We expect the entropy of the entire fluid to increase, not the entropy of individual fluid elements. Landau and Lifshitz write

$$\frac{d}{dt} \left[\int_{\mathcal{V}} d\mathcal{V} \rho s \right] = \int_{\mathcal{V}} d\mathcal{V} \left[\frac{\partial \rho s}{\partial t} \right] \tag{34}$$

Then

$$\frac{\partial \rho s}{\partial t} = s \frac{\partial \rho}{\partial t} + \rho \frac{\partial s}{\partial t}$$

$$= s \left(-(\nabla \cdot \rho \boldsymbol{v}) \right) + \rho \left[-\boldsymbol{v} \cdot \nabla s + \frac{1}{\rho T} \left(\nabla \cdot (\kappa \nabla T) + \frac{\eta}{2} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_i}{\partial x_i} \right)^2 + \zeta (\nabla \cdot \boldsymbol{v})^2 \right) \right]$$

$$= -\nabla \cdot (\rho s \boldsymbol{v}) + \frac{1}{T} \left(\nabla \cdot (\kappa \nabla T) + \frac{\eta}{2} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_i}{\partial x_i} \right)^2 + \zeta (\nabla \cdot \boldsymbol{v})^2 \right) \tag{35}$$

Consider the volume integral of this expression

$$\int_{\mathcal{V}} d\mathcal{V} \left[\frac{\partial \rho s}{\partial t} \right] =$$

$$-\int_{\mathcal{V}} d\mathcal{V} \nabla \cdot (\rho s \, \boldsymbol{v}) + \int_{\mathcal{V}} d\mathcal{V} \frac{\nabla \cdot (\kappa \nabla T)}{T} + \int_{\mathcal{V}} d\mathcal{V} \frac{\eta}{2T} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_i}{\partial x_i} \right)^2 +$$

$$\int_{\mathcal{V}} d\mathcal{V} \frac{\zeta}{T} (\nabla \cdot \boldsymbol{v})^2$$

Note

$$\int_{\mathcal{V}} d\mathcal{V} \nabla \cdot (\rho s \, \boldsymbol{v}) = \int_{\mathcal{A}} d\mathcal{A} \, \rho s \, \boldsymbol{v} \cdot \boldsymbol{1}_n \tag{37}$$

If the surface is removed, to infinity $\rho \, s \, \vec{v} \to 0$.

Also note

$$
\begin{aligned}
\nabla \cdot \left(\frac{\nabla T}{T} \right) &= \frac{1}{T} \nabla \cdot (\nabla T) + \nabla T \cdot \nabla \left(\frac{1}{T} \right) \\
&= \frac{1}{T} \nabla \cdot (\nabla T) + \nabla T \cdot \left(-\frac{1}{T^2} \nabla T \right) \\
&= \frac{1}{T} \nabla \cdot (\nabla T) - \frac{(\nabla T)^2}{T^2}
\end{aligned}
\tag{38}
$$

So, if $\nabla T \to 0$ as the $\mathcal{V}$ expands to ∞

$$
\begin{aligned}
\int_{\mathcal{V}} d\mathcal{V} \frac{\nabla \cdot (\kappa \nabla T)}{T} &= \kappa \int_{\mathcal{V}} d\mathcal{V} \nabla \cdot \left(\frac{\nabla T}{T} \right) + \kappa \int_{\mathcal{V}} d\mathcal{V} \frac{(\nabla T)^2}{T^2} \\
&= \kappa \int_{\mathcal{A}} d\mathcal{A} \left(\frac{\nabla T}{T} \right) \cdot \boldsymbol{1}_n + \kappa \int_{\mathcal{V}} d\mathcal{V} \frac{(\nabla T)^2}{T^2} \\
&= \kappa \int_{\mathcal{V}} d\mathcal{V} \frac{(\nabla T)^2}{T^2}
\end{aligned}
\tag{39}
$$

Using these results

$$\int_{\mathcal{V}} d\mathcal{V} \left[\frac{\partial \rho s}{\partial t} \right] = \kappa \int_{\mathcal{V}} d\mathcal{V} \frac{(\nabla T)^2}{T^2} + \int_{\mathcal{V}} d\mathcal{V} \frac{\eta}{2T} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_i}{\partial x_i} \right)^2 + \int_{\mathcal{V}} d\mathcal{V} \frac{\zeta}{T} (\nabla \cdot \boldsymbol{v})^2 \tag{40}$$

Finally

$$\frac{\partial \rho s}{\partial t} = \kappa \frac{(\nabla T)^2}{T^2} + \frac{\eta}{2T} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_i}{\partial x_i} \right)^2 + \frac{\zeta}{T} (\nabla \cdot \boldsymbol{v})^2 \tag{41}$$

This form illustrates that entropy production is positive for both thermal conductivity (∇T^2), and viscosity $(\eta > 0, \zeta > 0$ and squared quantities).

$$\frac{\partial \rho s}{\partial t} = \kappa \frac{(\nabla T)^2}{T^2} + \frac{\eta}{2 T} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_i}{\partial x_i} \right)^2 + \frac{\zeta}{T} (\nabla \cdot \boldsymbol{v})^2$$

References

Detailed derivation of the entropy generation and transport equation.
https://web.stanford.edu/~cantwell/AA210A_Course_Material/AA210A_Course_Notes/AA210_Fundamentals_of_Compressible_Flow_Ch_07_BJ_Cantwell.pdf

More basic discussion of the hydrodynamic equations, especially of concepts related to the internal energy of a flowing fluid.
http://www2.mpia-hd.mpg.de/~dullemon/lectures/fluiddynamics08/chap_1_hydroeq.pdf

The best derivation of the entropy equation is in Landau and Lifshitz (I downloaded a copy)
See also
http://astro.pas.rochester.edu/~aquillen/ast242/lecturenotes3.pdf